Group-invariant soliton equations and bi-Hamiltonian geometric curve flows in Riemannian symmetric spaces

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Abstract

Universal bi-Hamiltonian hierarchies of group-invariant (multicomponent) soliton equations are derived from non-stretching geometric curve flows $\gamma(t, x)$ in Riemannian symmetric spaces $M = G/H$, including compact semisimple Lie groups $M = K$ for $G = K \times K$, $H = \text{diag } G$. The derivation of these soliton hierarchies utilizes a moving parallel frame and connection 1-form along the curve flows, related to the Klein geometry of the Lie group $G \supset H$ where H is the local frame structure group. The soliton equations arise in explicit form from the induced flow on the frame components of the principal normal vector $N = \nabla_x \gamma_x$ along each curve, and display invariance under the equivalence subgroup in H that preserves the unit tangent vector $T = \gamma_x$ in the framing at any point x on a curve. Their bi-Hamiltonian integrability structure is shown to be geometrically encoded in the Cartan structure equations for torsion and curvature of the parallel frame and its connection 1-form in the tangent space $T_\gamma M$ of the curve flow. The hierarchies include group-invariant versions of sine–Gordon (SG) and modified Korteweg–de Vries (mKdV) soliton equations that are found to be universally given by curve flows describing non-stretching wave maps and mKdV analogs of non-stretching Schrödinger maps on G/H. These results provide a geometric interpretation and explicit bi-Hamiltonian formulation for many known multicomponent soliton equations. Moreover, all examples of group-invariant (multicomponent) soliton equations given by the present geometric framework can be constructed in an explicit fashion based on Cartan's classification of symmetric spaces.

1. Introduction and overview

The theory of integrable soliton equations displays many deep links to differential geometry, particularly as found in the study of geometric curve flows. In this paper, group-invariant soliton equations and their bi-Hamiltonian integrability structure are derived from studying non-stretching flows of curves in symmetric spaces G/H. Such spaces describe curved G-invariant Riemannian manifolds, generalizing the classical two-dimensional Riemannian geometries i.e., the sphere $S^2 \simeq SO(3)/SO(2)$, the hyperbolic plane $H^2 \simeq SL(2, \mathbb{R})/SO(2)$, and the Euclidean plane $\mathbb{R}^2 \simeq \text{Euc}(2)/SO(2)$, which are characterized by constant (positive, negative, zero) curvature.

Geometric curve flows in constant curvature spaces $M = S^2, H^2, \mathbb{R}^2$ are well known to yield [14,16,22,18] the mKdV hierarchy of scalar soliton equations and its hereditary recursion operator. The starting point is to formulate

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the flow of a non-stretching (inextensible) curve $\gamma(t, x)$ as geometrically given by $\gamma_t = h_\perp N + h_\parallel T$ in an adapted moving frame $(T, N) = (\gamma_x, *\gamma_x)$ (i.e. a Frenet frame) along the curve, where x is arclength and $*$ is the Hodge-star operator on vectors in the tangent plane $\mathbb{R}^2 \simeq T_x M$. Preservation of the non-stretching condition $\gamma_x \cdot \gamma_x = 1$ under the curve flow yields the relation $\partial_x h_\parallel = \kappa h_\perp$, allowing the tangential component $h_\parallel$ of the flow to be expressed in terms of the normal component $h_\perp$ and the curvature invariant $\kappa = N \cdot \nabla_x T$ of the curve. The flow equation combined with the Serret–Frenet structure equations of the moving frame then determine an evolution of this curvature invariant $\kappa_t = \mathcal{R}(h_\perp) + \chi h_\perp$ where $\mathcal{R} = D_x^2 + \kappa^2 + \kappa_x D_x^{-1} \kappa$ is seen to be the hereditary recursion operator of the mKdV hierarchy of soliton equations [29], and χ is the Gaussian curvature $(+1, -1, 0)$ of M. In particular, the flow given by $h_\perp = \kappa_x$, $h_\parallel = \frac{1}{2}\kappa^2$ yields the mKdV evolution equation on κ to within a convective term, $\kappa_t - \chi\kappa_x = \kappa_{3x} + \frac{3}{2}\kappa^2\kappa_x$ (called the $+1$ flow in the hierarchy). Higher order mKdV evolution equations arise from linear combinations of flows $h_\perp = \mathcal{R}^n(\kappa_x)$, $h_\parallel = D_x^{-1}(\kappa \mathcal{R}^n(\kappa_x))$, $n = 1, 2, \ldots$. The entire mKdV hierarchy $\kappa_t - \chi\kappa_x = \mathcal{R}^n(\kappa_x)$ thus sits in the class of geometric flows in which the evolution of the curve in the normal direction is a function $h_\perp = h_\perp(\kappa, \kappa_x, \ldots)$ of the differential invariants of the curve [25,26]. Such flows correspond to geometric equations $\gamma_t = f(\gamma_x, \nabla_x\gamma_x, \ldots)$ satisfied by the curve $\gamma(t, x)$.

It is less widely known that the geometric curve equations produced in this manner from the mKdV hierarchy of evolutions on κ are close analogs of Schrödinger maps $\gamma_t = J(\gamma_x)$, $J = \mathrm{i}\nabla_x$, on $S^2 \simeq \mathbb{C}P^1$ (identifying i with $*$ after complexification of the tangent plane). For instance the mKdV evolution itself yields $\gamma_t = \kappa_x N + \frac{1}{2}\kappa^2 T = \nabla_x(\kappa \ *\gamma_x) + \frac{3}{2}\kappa^2\gamma_x$ describing [1] a geometric curve equation $\gamma_t = K_\gamma(\gamma_x)$, $K_\gamma = \nabla_x^2 + \frac{3}{2}|\nabla_x\gamma_x|^2$ as obtained through the Serret–Frenet equations $\nabla_x T = \kappa N$, $\nabla_x N = -\kappa T$. This flow operator K_γ is the mKdV analog of a modified Schrödinger operator $J_\gamma = \mathrm{i}\nabla_x + (\arg\gamma_x)_x$ that preserves $|\gamma_x| = 1$. One geometrical difference between these operators is that $\bar{\gamma}_x \cdot J_\gamma(\gamma_x) = 0$ whereas $\bar{\gamma}_x \cdot K_\gamma(\gamma_x) = \frac{1}{2}\kappa^2 > 0$, implying (by dimensional considerations) that the non-stretching Schrödinger flow of the curve $\gamma(t, x)$ in S^2 is actually stationary $\gamma_t = J_\gamma(\gamma_x) = 0$ (so thus $\kappa_t = 0$), while in contrast the non-stretching mKdV flow $\gamma_t = K_\gamma(\gamma_x)$ is dynamical.

A related geometric evolution [1] comes from the kernel of the mKdV recursion operator $\mathcal{R}(h_\perp) = 0$ giving the evolution equation $\kappa_t = \chi h_\perp$ (called the -1 flow in the mKdV hierarchy), with $h_\perp$ satisfying $\partial_x h_\perp = -\kappa h_\parallel$, $\partial_x h_\parallel = \kappa h_\perp$. This flow obeys the conservation law $0 = \partial_x(h_\parallel^2 + h_\perp^2) = \nabla_x(\gamma_t \cdot \gamma_t)$ which implies it is conformally equivalent (under rescalings of t) to a flow with uniform speed $|\gamma_t| = \sqrt{h_\parallel^2 + h_\perp^2} = 1$. The evolution of the curve is then given by $h_\perp = \sin(\theta)$, $h_\parallel = \cos(\theta)$ in terms of the nonlocal invariant $\theta = -\int\kappa\,\mathrm{d}x$ of γ, yielding the sine–Gordon equation $\theta_{tx} = -\chi\sin(\theta)$ induced from the evolution on κ [21,28]. Equivalently, the relations $\theta = \arcsin(\kappa_t/\chi)$ and $\theta_x = -\kappa$ yield a hyperbolic equation $\kappa_{tx} = -\kappa\sqrt{\chi^2 - \kappa_t^2}$ on κ. The corresponding geometric curve equation satisfied by $\gamma(t, x)$ is $\nabla_x\gamma_t = \nabla_x(\sin(\theta)N) + \nabla_x(\cos(\theta)T) = 0$ after the Serret–Frenet equations are used. This curve flow is recognized [1] to be a uniform speed, non-stretching wave map $\nabla_x\gamma_t = \nabla_t\gamma_x = 0$ on S^2, with t, x viewed as light cone coordinates for the wave operator.

Moreover, the geometry of the two-dimensional surfaces swept out by all these non-stretching curve flows $\gamma(t, x)$ in $M = S^2$, H^2, $\mathbb{R}^2$ turns out to encode the bi-Hamiltonian integrability structure [30] of the mKdV soliton hierarchy. The relevant geometry [1] is given by the Riemannian connections ∇_t, ∇_x in the flow direction γ_t and tangent direction γ_x, specifically that they have vanishing torsion $\nabla_t\gamma_x - \nabla_x\gamma_t = [\gamma_t, \gamma_x] = 0$ and carry constant curvature $[\nabla_t, \nabla_x] = R(\gamma_t, \gamma_x) = \chi(\gamma_t \cdot *\gamma_x)*$ where $R(X, Y)$ is the Riemann curvature tensor of M with X, Y in the tangent plane. A projection of these equations into the normal space of γ yields $\kappa_t = \mathcal{H}(\varpi) + \chi h_\perp$, $\varpi = \mathcal{J}(h_\perp)$, expressed in terms of the flow invariant $\varpi = N \cdot \nabla_t T$, where $\mathcal{H} = D_x$, $\mathcal{J} = D_x + \kappa D_x^{-1}\kappa$ are found to be respectively the mKdV Hamiltonian cosymplectic and symplectic operators for the hierarchy of mKdV evolutions on κ. In this setting $h_\perp\partial/\partial\kappa$ represents a Hamiltonian vector field, and $\varpi\,\mathrm{d}\kappa$ represents a variational covector field with $\varpi = \delta H/\delta\kappa$ holding for some Hamiltonian $H = H(\kappa, \kappa_x, \ldots)$. Compatibility of the operators $\mathcal{H}$, $\mathcal{J}$ implies that $\mathcal{R} = \mathcal{H} \circ \mathcal{J}$ and its adjoint $\mathcal{R}^* = \mathcal{J} \circ \mathcal{H}$ generate a hierarchy of commuting vector fields $h_\perp^{(n)} = \mathcal{R}^n(\kappa_x)$ and corresponding involutive covector fields $\varpi^{(n)} = \mathcal{R}^{*n}(\kappa)$, $n = 0, 1, 2, \ldots$, related by $h_\perp^{(n)} = \mathcal{H}(\varpi^{(n)})$, $\varpi^{(n+1)} = \mathcal{J}(h_\perp^{(n)})$. For $n \geq 1$ this hierarchy possesses a bi-Hamiltonian structure $h_\perp^{(n)} = \mathcal{H}(\delta H^{(n)}/\delta\kappa) = \mathcal{J}^{-1}(\delta H^{(n+1)}/\delta\kappa)$ where $\mathcal{J}^{-1}$ is the formal inverse of $\mathcal{J}$ defined on the x-jet space of κ, and with the local Hamiltonians $H^{(n)} = H^{(n)}(\kappa, \kappa_x, \ldots)$ determined by $\varpi^{(n)} = \delta H^{(n)}/\delta\kappa$. Thus $\mathcal{H}$, $\mathcal{J}^{-1}$ are formally a bi-Hamiltonian pair of cosymplectic operators. Alternatively, $\mathcal{E} = \mathcal{H}\mathcal{J}\mathcal{H} = D_x^3 + \kappa^2 D_x + \kappa_x D_x^{-1}(\kappa D_x)$ provides an explicit cosymplectic operator compatible with $\mathcal{H}$, giving the bi-Hamiltonian structure $h_\perp^{(n)} = \mathcal{E}(\delta H^{(n-1)}/\delta\kappa) = \mathcal{H}(\delta H^{(n)}/\delta\kappa)$.

The bottom of the mKdV hierarchy $h_\perp{}^{(0)} = \kappa_x$, $\varpi^{(0)} = \kappa$, $H^{(0)} = \frac{1}{2}\kappa^2$ originates geometrically from the x-translation vector field $\kappa_x \partial/\partial\kappa$ which is a symmetry of the operators $\mathcal{H}$, $\mathcal{J}$. In terms of the jet space variables $(x, \kappa, \kappa_x, \ldots)$, the entire hierarchy has the form of homogeneous polynomials with non-zero scaling weight under the mKdV scaling symmetry $x \to \lambda x$, $\kappa \to \lambda^{-1}\kappa$. As a consequence, through a scaling formula (cf. [2]) the Hamiltonians are given by $H^{(n)} \equiv \frac{1}{1+2n}(\kappa + x\kappa_x)\varpi^{(n)} \equiv \frac{1}{1+2n}D_x^{-1}(\kappa h_\perp{}^{(n)}) = \frac{1}{1+2n}h_\parallel{}^{(n)}$ (modulo total x-derivatives) for $n \geq 0$, where $\partial_x h_\parallel{}^{(n)} = \kappa h_\perp{}^{(n)}$. Thus $\int h_\parallel{}^{(n)}dx$ represents a (scaled) Hamiltonian functional on the x-jet space of κ. The same expression carries over to the -1 flow in the mKdV hierarchy, $h_\perp{}^{(-1)} = \sin(\theta)$, $h_\parallel{}^{(-1)} = \cos(\theta)$, given in terms of $\kappa = -\theta_x$. Specifically, $h_\perp{}^{(-1)} = \mathcal{H}(\delta H^{(-1)}/\delta\kappa)$ holds for the Hamiltonian $H^{(-1)} = -h_\parallel{}^{(-1)}$, with $\varpi^{(-1)} = \delta H^{(-1)}/\delta\kappa = D_x^{-1}(\delta H^{(-1)}/\delta\theta)$.

Recent work [3] has generalized these results to the two known vector versions of the mKdV equation [34,6], by deriving their bi-Hamiltonian integrability structure along with their associated hierarchies of (higher order) vector soliton equations from geometric curve flows in the Riemannian symmetric spaces $SO(N+1)/SO(N) \simeq S^n$ and $SU(N)/SO(N)$. These two spaces exhaust (cf. [19]) all examples of irreducible symmetric spaces $G/SO(N)$ given by compact simple Lie groups G, and in the case $N = 2$ they both coincide with the classical Riemannian spherical geometry $S^2 \simeq SO(3)/SO(2) \simeq SU(2)/SO(2)$.

This geometric derivation of the vector mKdV hierarchies involves several main ideas. Firstly, for a non-stretching curve γ, the components of the principal normal in an adapted moving frame along γ play the role of a natural Hamiltonian flow variable. Secondly, the torsion and curvature equations of the Riemannian connections ∇_t, ∇_x on the two-dimensional surface of any flow $\gamma(t, x)$ of such curves in $G/SO(N)$ geometrically encode Hamiltonian cosymplectic and symplectic operators $\mathcal{H}$, $\mathcal{J}$, where x is the arclength on γ. In particular, the Hamiltonian structure looks simplest if the moving frame along γ is chosen so that it has a parallel connection matrix [9] given by an algebraic version of the basic geometric property $\nabla_x N \parallel T$, $\nabla_x T \perp T$ of an adapted moving frame generalized from S^2 to $G/SO(N)$. For such a moving parallel frame the torsion and curvature equations reduce to a simple form which can be derived directly from the infinitesimal Klein geometry of the tangent spaces $\mathfrak{g}/\mathfrak{so}(N)$ in $G/SO(N)$, where $\mathfrak{g}$ is the Lie algebra of G. Thirdly, within the class of non-stretching curves whose parallel moving frame is preserved by an $O(N-1)$ isotropy subgroup of the local frame structure group $SO(N)$ of the frame bundle of $G/SO(N)$, the encoded operators $\mathcal{H}$, $\mathcal{E} = \mathcal{H}\mathcal{J}\mathcal{H}$ comprise an $O(N-1)$-invariant bi-Hamiltonian pair. Moreover, they possess the mKdV scaling symmetry and exhibit symmetry invariance under x-translations. As a result, these operators yield a hereditary recursion operator $\mathcal{R} = \mathcal{H}\mathcal{J}$ generating a hierarchy of integrable curve flows in $G/SO(N)$, organized by their scaling weight, in which the flow variable satisfies an $O(N-1)$-invariant vector evolution equation. The $+1$ flow in the respective hierarchies for $G = SO(N+1)$, $SU(N)$ is given by the two known vector mKdV soliton equations [34], and there is a -1 flow described by vector hyperbolic equations [6] (variants of vector SG equations) coming from the kernel of the respective mKdV recursion operators in the two hierarchies. All the flows in both hierarchies correspond to commuting bi-Hamiltonian vector fields in the x-jet space of the flow variable. Finally, the evolution equations of the curve $\gamma(t, x)$ produced by the ± 1 flows in each hierarchy are G-invariant generalizations of the geometric map equations found for these flows in the case $S^2 \simeq SO(3)/SO(2) \simeq SU(2)/SO(2)$, identified as wave maps and mKdV analogs of Schrödinger maps, on $G/SO(N)$.

More recently there has been an extension of these results [4] deriving complex generalizations of the vector mKdV hierarchies along with their bi-Hamiltonian integrability structure from geometric curve flows in the Lie groups $G = SO(N+1)$, $SU(N)$. The derivation adapts the moving parallel frame formulation of non-stretching curve flows to these Lie groups by viewing them as Riemannian symmetric spaces in the standard manner [20,19], with the frame invariance group being a natural isotropy subgroup $U(N-1) \subset G$ contained in the local structure group of the frame bundle of $G = SO(N+1)$, $SU(N)$. This leads to bi-Hamiltonian hierarchies of curve flows $\gamma(t, x)$ in which the flow variable (again described by the components of the principal normal vector along the curves) obeys a $U(N-1)$-invariant vector evolution equation. In particular, the hierarchies contain the known complex generalizations [34] of the two versions of vector mKdV soliton equations, as well as complex generalizations of the corresponding vector SG equations which were first obtained by symmetry–integrability classifications [6].

A full generalization of these ideas and results will be presented here to obtain bi-Hamiltonian hierarchies of group-invariant soliton equations arising from geometric curve flows in general Riemannian symmetric spaces G/H. All irreducible examples of these spaces divide into two types, where either G is a simple Lie group (and H is a

compact subgroup invariant under an involutive automorphism of G) or G is a Lie group product $K \times K$ (and H is a diagonal subgroup) such that $G/H \simeq K$ is a compact simple Lie group.

Theorem 1.1. *For each Riemannian symmetric space G/H there is a family of bi-Hamiltonian hierarchies of non-stretching curve flows described by G-invariant geometric maps on G/H. The components of the principal normal vector along these curves in a moving parallel frame satisfy group-invariant (multicomponent) soliton equations of mKdV and SG type, with an explicit bi-Hamiltonian structure. In each hierarchy the geometric maps corresponding to the SG flow and the lowest order mKdV flow are universally given by a non-stretching wave map and a mKdV analog of a non-stretching Schrödinger map.*

Remark 1.2. The invariance group of the soliton equations and their bi-Hamiltonian integrability structure consists of the linear isotropy subgroup in $\mathrm{Ad}(H)$ that preserves the tangent vector for the corresponding curves in the tangent space $T_o G/H = \mathfrak{g}/\mathfrak{h}$ at the origin o. (By G-invariance, any geometric curve flow is equivalent to one that passes through the origin o in G/H.)

Remark 1.3. Up to isomorphism, the distinct bi-Hamiltonian hierarchies in the family admitted by a given Riemannian symmetric space G/H are in one-to-one correspondence with equivalence classes of unit-norm elements contained in any fixed maximally abelian subspace in $\mathfrak{g}/\mathfrak{h}$ modulo the Weyl group $W \subset \mathrm{Ad}(H)$.

These results provide a geometric origin and unifying interpretation that encompasses many examples of multicomponent soliton equations. All examples can, moreover, be written down in explicit form, using Cartan's classification of symmetric spaces [19].

To begin, some preliminaries are stated in Section 2 on the relevant differential geometric and Lie algebraic properties of symmetric spaces [19,20], particularly the infinitesimal Klein geometry [33] attached to $\mathfrak{g}/\mathfrak{h}$, and the Cartan subspaces in $\mathfrak{g}/\mathfrak{h}$ determined by the Lie algebras $\mathfrak{h}$, $\mathfrak{g}$ of H, G. These properties are used in an essential way in Section 3 for the study of non-stretching curve flows $\gamma(t, x)$ in G/H, where x is arclength (so $|\gamma_x| = 1$ is the non-stretching property using the Riemannian metric on G/H). The main starting point is given by the pullback of the torsion and curvature equations of the Riemannian connection on G/H to the two-dimensional surface of the flow of γ, as derived directly by a frame formulation in terms of the infinitesimal Klein geometry of $\mathfrak{g}/\mathfrak{h}$. Based on the decomposition of $\mathfrak{g}$ relative to $\mathrm{ad}(e)$ where e is a unit vector belonging to a Cartan subspace in $\mathfrak{g}/\mathfrak{h}$, a natural algebraic construction is given for a parallel moving frame along γ, evolving under the flow. Then in Section 4, the moving frame components of the torsion and curvature are shown to encode a bi-Hamiltonian pair of cosymplectic and symplectic operators $\mathcal{H}$, $\mathcal{J}$ with respect to a Hamiltonian flow variable given by the moving frame components of the principal normal along γ. The proof of this main result will involve extending the standard theory of bi-Hamiltonian structures [29,15] to the setting of Lie-algebra-valued variables.

The hierarchy of bi-Hamiltonian commuting flows produced by the operators $\mathcal{H}$, $\mathcal{J}$ is studied in Section 5, where the $+1$ and -1 flows are constructed and found to yield group-invariant multicomponent versions of mKdV and SG equations associated to all irreducible Riemannian symmetric spaces. The geometric map equations produced by these universal ± 1 flows are derived in detail. Concluding remarks on extensions of these results will be made in Section 6.

Some related results of interest have been obtained in recent literature. In one direction, Hamiltonian operators for one of the vector mKdV equations as well as for a related scalar–vector KdV system have been derived from non-stretching curve flows in N-dimensional constant curvature Riemannian geometries (i.e. the N-sphere) and in N-dimensional flat conformal geometries (i.e. the Möbius N-sphere), using two different approaches. One approach [30,31] viewed these geometries as locally modeled by Klein geometries on $\mathbb{R}^N$ defined respectively by a Euclidean isometry group action $\mathrm{Euc}(N)$ and a Möbius (conformal) isometry group action $\mathrm{Mob}(N) \simeq SO(N+1, 1)$. A subsequent derivation [1,5] used just the intrinsic Riemannian and conformal connections defined on the underlying manifold S^N. Both derivations utilize the approach of studying curve flows via parallel moving frames, first introduced in [22,27,30]. The same approach has also been applied recently [36] to curve flows in symplectic geometry $S_p(N+1)/S_p(N) \times S_p(1)$, giving a geometric derivation of a quaternionic (non-commutative) mKdV equation.

Earlier fundamental work appeared in [7,8] on multicomponent mKdV equations derived from symmetric spaces by algebraic and geometric considerations, and on investigation of Hamiltonian structures of such equations.

Another direction [10,11] has concentrated on deriving the known scalar soliton equations from non-stretching curve flows in various classical plane geometries: hyperbolic plane, affine and fully affine plane, plane similarity

geometry. These geometries together with the Euclidean plane are characterized by their respective isometry groups $SL(2)$, $SA(2)$ and $A(2)$, Sim(2), Euc(2), each acting locally and effectively on $\mathbb{R}^2$; as such they comprise the main examples of planar Klein geometries $G/H \simeq \mathbb{R}^2$ where G is the isometry group and $H \subset G$ is a stabilizer subgroup that leaves fixed the origin in $\mathbb{R}^2$. In this geometric setting there is a natural group-invariant notion of geometric curve flows based on the evolution of differential invariants [29] of a non-stretching curve γ, formulated using a moving frame $(T, N) = (\gamma_x, \gamma_{xx})$ where x is a natural G-invariant arclength on γ. Generalizations to non-stretching flows of space curves in Klein geometries on $\mathbb{R}^3$ have also been studied in a few cases [12,13]: Euclidean space Euc(3), similarity geometry Sim(3), affine space $A(3)$, centro-affine space $SL(3)$. The results obtained to date have provided an elegant geometric origin for many scalar soliton equations. For instance, the KdV equation and the Sawada–Kotera equation are noted to arise respectively as a centro-affine and affine version of the mKdV equation coming from Euclidean geometry. It should be possible to derive the bi-Hamiltonian integrability structure of these solitons equations by adapting the parallel moving frame approach used in the present work to the setting of affine geometry and similarity geometry.

2. Preliminaries

A symmetric space $M = G/H$ is *irreducible* if it is not a product of smaller symmetric spaces. All irreducible Riemannian symmetric spaces have a classification into two types [20,19]:

(I) G is a connected simple Lie group and $H \subset G$ is a compact Lie subgroup invariant under an involutive automorphism σ of G (in this case M is compact or noncompact according to whether G is).

(II) G is a product $K \times K$ for a connected compact simple Lie group K and H is its diagonal product subgroup in G on which a permutation of the product factors acts as an involutive automorphism σ, with $G/H \simeq K \simeq H$ (in this case, since G is compact, so is M).

Henceforth let $M = G/H$ be an irreducible Riemannian symmetric space in which G acts effectively on the manifold M by left multiplication. The automorphism σ induces a decomposition on the Lie algebra of G, given by a direct sum of vector spaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $\sigma(\mathfrak{h}) = \mathfrak{h}$, $\sigma(\mathfrak{m}) = -\mathfrak{m}$, with Lie bracket relations $[\mathfrak{h}, \mathfrak{h}]_{\mathfrak{g}} \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}]_{\mathfrak{g}} \subset \mathfrak{m}$, $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{g}} \subset \mathfrak{h}$, where $\mathfrak{h}$ is identified with the Lie algebra of H and $\mathfrak{m} = T_o M$ is identified with the tangent space at the origin o in M. The group action of G on M provides a canonical isomorphism $T_x M \simeq_G \mathfrak{m}$.

2.1. Riemannian metric and connection

The Riemannian structure of G/H comes from the Cartan–Killing form $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$. Since $\mathfrak{g}$ is a simple Lie algebra, the subspaces $\mathfrak{h}, \mathfrak{m}$ are orthogonal $\langle \mathfrak{h}, \mathfrak{m} \rangle = 0$ and hence the Cartan–Killing form restricts to yield a nondegenerate inner product on $\mathfrak{m}$ which is $\mathrm{Ad}_{\mathfrak{g}}(H)$ invariant. This inner product $\langle \cdot, \cdot \rangle_{\mathfrak{m}}$ has a definite sign, negative or positive, according to whether $\mathfrak{g}$ is compact or noncompact. In the noncompact case, the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is dual to $\mathfrak{g}^* = \mathfrak{h} \oplus \mathrm{i}\mathfrak{m}$ which defines a compact real Lie algebra contained in the complexification of $\mathfrak{g}$. This leads to a corresponding duality [20,19] between compact and noncompact Riemannian symmetric spaces of type I. As there are no noncompact Riemannian symmetric spaces of type II, we will henceforth restrict attention to the compact case in order to include both types I and II.

Under the canonical identification $\mathfrak{m} \simeq_G T_x M$, the negative-definite inner product $\langle \cdot, \cdot \rangle_{\mathfrak{m}}$ defines a Riemannian metric tensor g on M given by

$$g(X, Y) = -\langle X, Y \rangle_{\mathfrak{m}} \tag{1}$$

for all G-invariant vector fields X, Y in $T_x M$. Associated to this metric g is a unique torsion-free G-invariant Riemannian connection ∇, determined by $\nabla g = 0$, where

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0 \tag{2}$$

is the torsion-free property. Here $[\cdot, \cdot]$ denotes the commutator of vector fields on M. The Riemann curvature tensor of the connection ∇ is given by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = -\mathrm{ad}_{\mathfrak{m}}([X, Y]_{\mathfrak{m}}) \tag{3}$$

with $\mathrm{ad}_{\mathfrak{m}}$ viewed as a linear map on $T_x M \simeq_G \mathfrak{m}$. This curvature tensor is G-invariant and covariantly constant $\nabla R = 0$. Note that left multiplication by G acting on M generates the isometries of the manifold M, namely the isometry group of g is G.

Recall, the linear isotropy group H^* of G/H is the representation of H by linear transformations on $T_o M$ at the origin o in M. This group is isomorphic to $\mathrm{Ad}_{\mathfrak{m}}(H)$ through the identification $T_o M = \mathfrak{m}$.

2.2. Klein geometry and soldering frames

For any Riemannian symmetric space $M = G/H$, the Lie group G has the structure of a principal H-bundle over M, $\pi : G \to G/H$ [20,33]. Consider the Maurer–Cartan form w_G on G, which is a left-invariant $\mathfrak{g}$-valued 1-form that provides the canonical identification $T_e G = \mathfrak{g}$ where e is the identity element of G. The Maurer–Cartan form satisfies $\mathrm{d}w_G + \frac{1}{2}[w_G, w_G]_{\mathfrak{g}} = 0$ and hence, viewed geometrically, it is a zero-curvature connection on the bundle G. This collective structure is called [33] the Klein geometry of $M = G/H$. It has a useful reformulation locally on M involving the Lie algebra structure of the tangent spaces of G/H.

Let $\psi : U \to \pi^{-1}(U) \subset G$ be any local section of this bundle, where (U, ψ) are coordinates for a local trivialization $G \approx U \times H$. The tangent space of the trivialization $U \times H$ at any point $(x, \psi(x))$ is isomorphic to the Lie algebra $\mathfrak{g}$ of G. In particular, note $\pi^* : \mathfrak{g} \simeq \mathfrak{m} \times \mathfrak{h} \to \mathfrak{m}$ as vector spaces, with $T_x M \simeq_G \mathfrak{m} = \mathfrak{g}/\mathfrak{h}$, $T_{\psi(x)} H \simeq \mathfrak{h}$ and $T_{(x, \psi(x))} G \simeq \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. Now, the pullback of the Maurer–Cartan form w_G by ψ yields a $\mathfrak{g}$-valued 1-form $^{\mathfrak{g}}\omega$ (locally) on M, satisfying the Cartan structure equation

$$D\,^{\mathfrak{g}}\omega + \frac{1}{2}[^{\mathfrak{g}}\omega, \,^{\mathfrak{g}}\omega]_{\mathfrak{g}} = 0 \tag{4}$$

where D is the total exterior derivative operator on M and $[\cdot, \cdot]_{\mathfrak{g}}$ is understood to denote the Lie algebra bracket in $\mathfrak{g}$ composed with the wedge product in $T_x^* M$. (Informally, we can view $^{\mathfrak{g}}\omega : T_x M \simeq_G \mathfrak{m} = \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ as attaching an infinitesimal Klein geometry to $\mathfrak{g}/\mathfrak{h}$.) Note a change of the local section ψ, described by a smooth function $h : U \to H$ yielding the local section $\tilde{\psi} = \psi h$, induces a change of $^{\mathfrak{g}}\omega$ given by

$$\widetilde{^{\mathfrak{g}}\omega} = \mathrm{Ad}(h^{-1})\,^{\mathfrak{g}}\omega + h^{-1} Dh. \tag{5}$$

This is called [33] a gauge transformation on $^{\mathfrak{g}}\omega$ relative to the gauge ψ.

Introduce

$$e := \frac{1}{2}(^{\mathfrak{g}}\omega - \sigma(^{\mathfrak{g}}\omega)), \qquad \omega := \frac{1}{2}(^{\mathfrak{g}}\omega + \sigma(^{\mathfrak{g}}\omega)), \tag{6}$$

denoting the even and odd parts of $^{\mathfrak{g}}\omega$, defined using the automorphism σ of $\mathfrak{g}$. The corresponding decomposition under a gauge transformation on $^{\mathfrak{g}}\omega$ yields

$$\tilde{e} = \mathrm{Ad}(h^{-1})e, \qquad \tilde{\omega} = \mathrm{Ad}(h^{-1})\omega + h^{-1} Dh. \tag{7}$$

Proposition 2.1. *On $U \subset M$, the $\mathfrak{m}$-valued 1-form e represents a linear coframe and the $\mathfrak{h}$-valued 1-form ω represents a linear connection, with H as the structure (gauge) group defining a bundle of linear frames.*

Let the 2-forms

$$\mathfrak{T} = De + [\omega, e]_{\mathfrak{g}} = \mathcal{D}e \quad \text{and} \quad \mathfrak{R} = D\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{h}} = [\mathcal{D}, \mathcal{D}] \tag{8}$$

denote the $\mathfrak{m}$-valued torsion and $\mathfrak{h}$-valued curvature associated to the linear connection and coframe, where

$$\mathcal{D} = D + [\omega, \cdot]_{\mathfrak{g}} \tag{9}$$

is the (gauge) covariant exterior derivative.

Proposition 2.2. *The Cartan structure equation* (4) *decomposes with respect to σ to give*

$$\mathfrak{T} = 0, \quad \mathfrak{R} = -\frac{1}{2}[e, e]_{\mathfrak{m}}, \tag{10}$$

so thus $\mathcal{D}$ is torsion-free and has covariantly constant curvature.

The precise interpretation of e and ω in these propositions comes from a basis expansion $e = e_a \mathbf{m}^a$, $\omega = \omega_i \mathbf{h}^i$, where $\mathbf{m}^a$, $\mathbf{h}^i$ are any fixed orthonormal basis for $\mathfrak{m}$, $\mathfrak{h}$ with respect to the Cartan–Killing inner product,

$$\langle \mathbf{m}^a, \mathbf{m}^b \rangle = -\delta^{ab}, \quad \langle \mathbf{h}^i, \mathbf{h}^j \rangle = -\delta^{ij}, \quad \langle \mathbf{m}^a, \mathbf{h}^i \rangle = 0. \tag{11}$$

Then $\{e_a\}$ will define frame covectors spanning the cotangent space of M at x, and $\{\omega_i\}$ will define connection 1-forms related to the exterior derivative of the frame covectors by the torsion-free property of the covariant derivative $\mathcal{D}$ on M at x.

Most importantly, e and ω provide a natural soldering of this frame geometry of $M = G/H$ locally onto the Riemannian geometry of $M = G/H$. Let e^* be dual to e, namely a $\mathfrak{m}$-valued vector such that $-\langle e^*, e \rangle_{\mathfrak{m}} = \mathrm{id}_x$ is the identity map on $T_x M$. Then e^* represents an orthonormal linear frame on $U \subset M$, so thus $\{e_a^*\}$ as given by a basis expansion $e^* = e_a^* \mathbf{m}^a$ will define frame vectors spanning the tangent space of M at x and obeying the orthonormality property $e_a^* \rfloor e_b = \delta_{ab}$.

Theorem 2.3. *The linear coframe e and linear connection ω determine a Riemannian metric and Riemannian connection on $M = G/H$ given by*

$$g = -\langle e, e \rangle_{\mathfrak{m}}, \quad \nabla \cdot = -\langle e^*, \mathcal{D}(e \rfloor \cdot) \rangle_{\mathfrak{m}} = -\langle e^*, D(e \rfloor \cdot) + [\omega, (e \rfloor \cdot)]_{\mathfrak{g}} \rangle_{\mathfrak{m}}. \tag{12}$$

In particular, for all X, Y in $T_x M$,

$$g(X, Y) = -\langle e_X, e_Y \rangle_{\mathfrak{m}} \tag{13}$$

and

$$\nabla_X Y = -\langle e^*, \mathcal{D}_X e_Y \rangle_{\mathfrak{m}}, \quad \text{or equivalently} \quad \mathcal{D}_X e_Y = e \rfloor \nabla_X Y \tag{14}$$

where $e_X = e \rfloor X$, $e_Y = e \rfloor Y$ are the coframe projections of X, Y. Moreover, the Riemannian curvature is determined by the curvature of the exterior covariant derivative $\mathcal{D}$,

$$R(X, Y)Z = -\langle e^*, [\mathcal{D}_X, \mathcal{D}_Y]e_Z - \mathcal{D}_{[X,Y]}e_Z \rangle_{\mathfrak{m}} = -\langle e^*, [\mathfrak{R}_{X,Y}, e_Z]_{\mathfrak{g}} \rangle_{\mathfrak{m}} \tag{15}$$

with (cf. Proposition 2.2)

$$\mathfrak{R}_{X,Y} = \mathfrak{R} \rfloor (X \wedge Y) = -[e_X, e_Y]_{\mathfrak{m}}. \tag{16}$$

Hence

$$R(X, Y) = \langle [[e_X, e_Y]_{\mathfrak{m}}, e]_{\mathfrak{g}}, e^* \rangle_{\mathfrak{m}}, \tag{17}$$

or equivalently

$$R(\cdot, \cdot) = -\langle [e] \cdot, e \rfloor \cdot]_{\mathfrak{m}}, [e^*, e]_{\mathfrak{m}} \rangle_{\mathfrak{h}}. \tag{18}$$

Here note $[e^*, e]_{\mathfrak{m}}$ defines a $\mathfrak{h}$-valued linear map on $T_x M$ displaying skew-symmetry with respect to g: $g(Y, [e^*, e]_{\mathfrak{m}} X) = [e_Y, e_X] = -g(X, [e^*, e]_{\mathfrak{m}} Y)$.

Remark. The torsion-free property of the Riemannian connection is expressed by (cf. Proposition 2.2)

$$T(X, Y) = -\langle e^*, (\mathcal{D}_Y e) \rfloor X - (\mathcal{D}_X e) \rfloor Y \rangle_{\mathfrak{m}} = -\langle e^*, \mathfrak{T}_{X,Y} \rangle_{\mathfrak{m}} = 0 \tag{19}$$

with

$$\mathfrak{T}_{X,Y} = \mathfrak{T} \rfloor (X \wedge Y). \tag{20}$$

The soldering structure of the Riemannian connection has the following alternative formulation.

Corollary 2.4. *For all X, Y in $T_x M$,*

$$\nabla_X e = [e, \omega_X]_{\mathfrak{g}}, \qquad [\nabla_X, \nabla_Y]e - \nabla_{[X,Y]}e = -e \rfloor R(X, Y) = [e, \mathfrak{R}_{X,Y}]_{\mathfrak{g}} \tag{21}$$

where $\omega_X = X \rfloor \omega$ is the linear connection projected in the X direction on M.

In examples, it will be convenient to work in a matrix representation for $\mathfrak{m}$, $\mathfrak{h}$, $\mathfrak{g}$ viewed as subspaces in $\mathfrak{gl}(N, \mathbb{R})$ for suitable $N \geq 1$. Hence $\mathbf{m}^a$, $\mathbf{h}^i$ will then represent $N \times N$ matrices satisfying the commutator relations

$$[\mathbf{h}^i, \mathbf{h}^j] = c^{ij}{}_k \mathbf{h}^k, \qquad [\mathbf{m}^a, \mathbf{m}^b] = c^{ab}{}_i \mathbf{h}^i, \qquad [\mathbf{h}^i, \mathbf{m}^a] = c^{ia}{}_b \mathbf{m}^b, \tag{22}$$

where $c^{ij}{}_k$, $c^{ab}{}_i c^{ia}{}_b$ are the structure constants of $\mathfrak{g}$ with respect to the matrix basis $\{\mathbf{m}^a, \mathbf{h}^i\}$. More directly, $\mathfrak{h}$ and $\mathfrak{g}$ can be represented as matrix subalgebras in $\mathfrak{gl}(N, \mathbb{R})$, with $\mathfrak{m}$ represented as the matrix subspace $\mathfrak{g}/\mathfrak{h}$ in $\mathfrak{gl}(N, \mathbb{R})$. Then the soldering structure is given by matrix formulas

$$g(X, Y) = -\chi \mathrm{tr}(e_X e_Y), \tag{23}$$
$$\nabla_X Y = -\chi \mathrm{tr}(e^* \mathcal{D}_X e_Y), \quad R(X, Y) = -\chi \mathrm{tr}([e^*, e][e_X, e_Y]), \tag{24}$$

for some constant χ depending on the choice of matrix representation of $\mathfrak{m}$, where $e_X = (e_a \rfloor X)\mathbf{m}^a$, $e_Y = (e_a \rfloor Y)\mathbf{m}^a$ are $N \times N$ matrices belonging to $\mathfrak{m}$; $e = e_a \mathbf{m}^a$, $e^* = e_a^* \mathbf{m}^a$ are $N \times N$ vector-valued matrices belonging to $T_x^* M \otimes \mathfrak{m}$, $T_x M \otimes \mathfrak{m}$ respectively. Moreover,

$$\nabla_X e_a^* = -c^{ib}{}_a(\omega_X)_i e_b^*, \tag{25}$$
$$R(X, Y)e_a^* = -c^{ib}{}_a(\mathfrak{R}_{X,Y})_i e_b^* = c^{cd}{}_i c^{ib}{}_a X_c Y_d e_b^*, \tag{26}$$

with $X_a = e_a \rfloor X$, $Y_a = e_a \rfloor Y$.

Due to the soldering, the frame covectors obey the orthonormality property $g(e_a^*, e_b^*)^{-1} = -\mathrm{tr}(\mathbf{m}^a \mathbf{m}^b)$.

2.3. Action of ad(e)

It is useful to consider an algebraic decomposition of $\mathfrak{g}$ with respect to ad(e) where e is the soldering frame. Let X be a vector in $T_x M$ and consider the element $e_X = e \rfloor X$ given by the frame projection of X into $\mathfrak{m} \subset \mathfrak{g}$. The centralizer of e_X is the set $\mathfrak{c}(e_X)$ of all elements annihilated by ad(e_X) in the Lie algebra $\mathfrak{g}$, namely $[e_X, \mathfrak{c}(e_X)] = 0$. This set has the obvious structure of a Lie subalgebra of $\mathfrak{g}$, since ad(e_X)$[\mathfrak{c}(e_X), \mathfrak{c}(e_X)] = 0$ from the fact that ad acts as a derivation on the Lie bracket of $\mathfrak{g}$. Write $\mathfrak{c}(e_X)_{\mathfrak{m}}$, $\mathfrak{c}(e_X)_{\mathfrak{h}}$ for the intersection of $\mathfrak{c}(e_X)$ with the subspaces $\mathfrak{m}$, $\mathfrak{h}$ in $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, and $\mathfrak{c}(e_X)_{\mathfrak{m}}^{\perp}$, $\mathfrak{c}(e_X)_{\mathfrak{h}}^{\perp}$ for their respective orthogonal complements (perp spaces) where

$$\mathfrak{m} = \mathfrak{c}(e_X)_{\mathfrak{m}} \oplus \mathfrak{c}(e_X)_{\mathfrak{m}}^{\perp}, \qquad \mathfrak{h} = \mathfrak{c}(e_X)_{\mathfrak{h}} \oplus \mathfrak{c}(e_X)_{\mathfrak{h}}^{\perp}. \tag{27}$$

Thus, note $[e_X, \mathfrak{c}(e_X)_{\mathfrak{m}}] = [e_X, \mathfrak{c}(e_X)_{\mathfrak{h}}] = 0$, while $[e_X, \mathfrak{c}(e_X)_{\mathfrak{m}}^{\perp}] \neq 0$, $[e_X, \mathfrak{c}(e_X)_{\mathfrak{h}}] \neq 0$.

Proposition 2.5. *The centralizer subspaces $\mathfrak{c}(e_X)_{\mathfrak{m}}$, $\mathfrak{c}(e_X)_{\mathfrak{h}}$ of e_X have the Lie bracket relations*

$$[\mathfrak{c}(e_X)_{\mathfrak{h}}, \mathfrak{c}(e_X)_{\mathfrak{h}}] \subseteq \mathfrak{c}(e_X)_{\mathfrak{h}}, \tag{28}$$
$$[\mathfrak{c}(e_X)_{\mathfrak{h}}, \mathfrak{c}(e_X)_{\mathfrak{m}}] \subseteq \mathfrak{c}(e_X)_{\mathfrak{m}}, \qquad [\mathfrak{c}(e_X)_{\mathfrak{m}}, \mathfrak{c}(e_X)_{\mathfrak{m}}] \subseteq \mathfrak{c}(e_X)_{\mathfrak{h}}. \tag{29}$$

Hence the perp spaces satisfy the relations

$$[\mathfrak{c}(e_X)_{\mathfrak{h}}, \mathfrak{c}(e_X)_{\mathfrak{h}}^{\perp}] \subseteq \mathfrak{c}(e_X)_{\mathfrak{h}}^{\perp}, \qquad [\mathfrak{c}(e_X)_{\mathfrak{m}}, \mathfrak{c}(e_X)_{\mathfrak{m}}^{\perp}] \subseteq \mathfrak{c}(e_X)_{\mathfrak{h}}^{\perp}, \tag{30}$$
$$[\mathfrak{c}(e_X)_{\mathfrak{h}}, \mathfrak{c}(e_X)_{\mathfrak{m}}^{\perp}] \subseteq \mathfrak{c}(e_X)_{\mathfrak{m}}^{\perp}, \qquad [\mathfrak{c}(e_X)_{\mathfrak{m}}, \mathfrak{c}(e_X)_{\mathfrak{h}}^{\perp}] \subseteq \mathfrak{c}(e_X)_{\mathfrak{m}}^{\perp}. \tag{31}$$

The proof uses just the Lie bracket structure of $\mathfrak{g}$ along with the invariance of the Cartan–Killing inner product. For instance, consider $[\mathfrak{c}(e_X)_\mathfrak{h}, \mathfrak{c}(e_X)_\mathfrak{h}]$. Because $\mathrm{ad}(e_X)$ is a derivation it annihilates $[\mathfrak{c}(e_X)_\mathfrak{h}, \mathfrak{c}(e_X)_\mathfrak{h}]$ and hence, since $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, $[\mathfrak{c}(e_X)_\mathfrak{h}, \mathfrak{c}(e_X)_\mathfrak{h}]$ is contained in $\mathfrak{c}(e_X)_\mathfrak{h}$. Thus $[\mathfrak{c}(e_X)_\mathfrak{h}, \mathfrak{c}(e_X)_\mathfrak{h}]$ is orthogonal to $\mathfrak{c}(e_X)_\mathfrak{h}^\perp$, and so by invariance, $0 = \langle \mathfrak{c}(e_X)_\mathfrak{h}^\perp, [\mathfrak{c}(e_X)_\mathfrak{h}, \mathfrak{c}(e_X)_\mathfrak{h}] \rangle_\mathfrak{h} = \langle \mathfrak{c}(e_X)_\mathfrak{h}, [\mathfrak{c}(e_X)_\mathfrak{h}, \mathfrak{c}(e_X)_\mathfrak{h}^\perp] \rangle_\mathfrak{h}$ which implies $[\mathfrak{c}(e_X)_\mathfrak{h}, \mathfrak{c}(e_X)_\mathfrak{h}^\perp]$ is contained in $\mathfrak{c}(e_X)_\mathfrak{h}^\perp$. The other relations are established similarly.

The observation that e_X lies in $\mathfrak{c}(e_X)_\mathfrak{m}$, combined with the Lie bracket relations in Proposition 2.5, shows that $\mathrm{ad}(e_X)$ maps $\mathfrak{c}(e_X)_\mathfrak{h}$ into $\mathfrak{c}(e_X)_\mathfrak{m}$, $\mathfrak{c}(e_X)_\mathfrak{h}^\perp$ into $\mathfrak{c}(e_X)_\mathfrak{m}^\perp$, and vice versa. Consequently, $\mathrm{ad}(e_X)^2$ is well defined as a linear mapping of the subspaces $\mathfrak{c}(e_X)_\mathfrak{h}^\perp$ and $\mathfrak{c}(e_X)_\mathfrak{m}^\perp$ into themselves. These subspaces are likewise taken into themselves by all the linear maps $\mathrm{ad}(\mathfrak{c}(e_X)_\mathfrak{h})$, corresponding to the infinitesimal action of the group of linear transformations that preserves e_X. In particular, let H_X^* be the closed subgroup in the linear isotropy group H^* given by $\mathrm{Ad}(h)e_X = e_X$. Since $[\mathrm{ad}(e_X), \mathrm{ad}(\mathfrak{c}(e_X)_\mathfrak{h})] = \mathrm{ad}([e_X, \mathfrak{c}(e_X)_\mathfrak{h}]) = 0$, note that $\mathfrak{c}(e_X)_\mathfrak{h}$ is isomorphic to the Lie algebra of H_X^* and that the linear transformations in this group commute with the linear map $\mathrm{ad}(e_X)^2$. Schur's lemma applied to $\mathrm{ad}(e_X)^2$ now gives the following result.

Proposition 2.6. (i) $\mathfrak{c}(e_X)_\mathfrak{m}^\perp$ and $\mathfrak{c}(e_X)_\mathfrak{h}^\perp$ *are isomorphic as vector spaces under the linear map* $\mathrm{ad}(e_X)$. (ii) $\mathfrak{c}(e_X)_\mathfrak{m}^\perp$ *and* $\mathfrak{c}(e_X)_\mathfrak{h}^\perp$ *each decompose into a direct sum of subspaces given by irreducible representations of the group* H_X^* *on which the linear map* $\mathrm{ad}(e_X)^2$ *is a multiple of the identity.*

Note $\dim \mathfrak{c}(e_X)_\mathfrak{m}^\perp = \dim \mathfrak{c}(e_X)_\mathfrak{h}^\perp = \dim \mathfrak{m} - \dim H_X^*$, so these spaces will be of minimal dimension (depending on e_X) precisely when the group H_X^* is of maximal size.

The decomposition of these vector spaces in Proposition 2.6 will be maximally irreducible whenever the linear map $\mathrm{ad}(e_X)^2$ has equal eigenvalues, whereby $\mathrm{ad}(e_X)^2 = \chi \, \mathrm{id}_\perp$ is a multiple of the identity map $\mathrm{id}_\perp$ on $\mathfrak{c}(e_X)_\mathfrak{m}^\perp \simeq \mathfrak{c}(e_X)_\mathfrak{h}^\perp$. In this situation, additional Lie bracket relations can be established as follows. Introduce the bracket $(\cdot, \cdot) := [\mathrm{ad}(e_X)\cdot, \cdot]$ on the centralizer subspace $\mathfrak{c}(e_X)_\mathfrak{g}^\perp := \mathfrak{c}(e_X)_\mathfrak{m}^\perp \oplus \mathfrak{c}(e_X)_\mathfrak{h}^\perp$ of e_X in $\mathfrak{g}$, and write $I_X := \mathrm{ad}(e_X)^2$. Since $\mathrm{ad}(e_X)$ acts as a derivation, note

$$I_X(y, z) = [\mathrm{ad}(e_X)I_X y, z] + [\mathrm{ad}(e_X)y, I_X z] + 2[I_X y, \mathrm{ad}(e_X)z]$$
$$= (I_X y, z) + (y, I_X z) - 2(z, I_X y) \tag{32}$$

for all vectors y, z in $\mathfrak{c}(e_X)_\mathfrak{g}^\perp$. Now suppose $I_X = \chi \, \mathrm{id}_\perp$, where χ is a non-zero constant. Then the identity (32) yields $\chi \, \mathrm{id}_\perp(y, z) = 2\chi(y, z) - 2\chi(z, y)$ and hence $(y, z)_\perp = 2(z, y)_\perp$ where a subscript $\perp$ denotes projection onto $\mathfrak{c}(e_X)_\mathfrak{g}^\perp$. This identity implies $(y, y)_\perp = 0$, and hence by polarization, $(y, z)_\perp = 0$. As a result, it follows that $0 = (y, z)_\perp - (z, y)_\perp = [\mathrm{ad}(e_X)y, z]_\perp + [y, \mathrm{ad}(e_X)z]_\perp = \mathrm{id}_\perp \mathrm{ad}(e_X)[y, z]$ whence $[y, z]_\perp = 0$ since $\mathrm{ad}(e_X)$ is nondegenerate on $\mathfrak{c}(e_X)_\mathfrak{g}^\perp$.

Proposition 2.7. *If* $\mathrm{ad}(e_X)^2 = \chi \, \mathrm{id}_\perp$ *is multiple of the identity on* $\mathfrak{c}(e_X)_\mathfrak{m}^\perp \simeq \mathfrak{c}(e_X)_\mathfrak{h}^\perp$, *then these centralizer subspaces obey the Lie bracket relations*

$$[\mathfrak{c}(e_X)_\mathfrak{h}^\perp, \mathfrak{c}(e_X)_\mathfrak{h}^\perp] \subseteq \mathfrak{c}(e_X)_\mathfrak{h}, \qquad [\mathfrak{c}(e_X)_\mathfrak{m}^\perp, \mathfrak{c}(e_X)_\mathfrak{m}^\perp] \subseteq \mathfrak{c}(e_X)_\mathfrak{h}, \tag{33}$$

$$[\mathfrak{c}(e_X)_\mathfrak{m}^\perp, \mathfrak{c}(e_X)_\mathfrak{h}^\perp] \subseteq \mathfrak{c}(e_X)_\mathfrak{m}. \tag{34}$$

2.4. Cartan spaces

The algebraic structure of the spaces $\mathfrak{c}(e_X)_\mathfrak{h}^\perp$ and $\mathfrak{c}(e_X)_\mathfrak{m}^\perp$ along with the group H_X^* will have a key role in the later construction of a parallel moving frame formulation for curve flows in Riemannian symmetric spaces. Part of this algebraic structure is effectively independent of e_X.

Let $\mathfrak{c}_0(e_X)$ denote the center of $\mathfrak{c}(e_X)$, which is an abelian subalgebra of $\mathfrak{g}$. Write $\mathfrak{c}_0(e_X)_\mathfrak{m}$, $\mathfrak{c}_0(e_X)_\mathfrak{h}$, for the intersections of $\mathfrak{c}_0(e_X)$ with $\mathfrak{m}$, $\mathfrak{h}$, and write $\mathfrak{a}$ for any maximal abelian subspace in $\mathfrak{m}$.

It can be shown [19] that the image of any maximal abelian subspace $\mathfrak{a}$ under the group H^* is the vector space $\mathrm{Ad}(H)\mathfrak{a} = \mathfrak{m}$. Hence e_X is contained in some such subspace, namely $\mathfrak{a} = \mathfrak{c}_0(e_X)_\mathfrak{m} \subseteq \mathfrak{c}(e_X)_\mathfrak{m} \subset \mathfrak{m}$. These subspaces $\mathfrak{a} \subset \mathfrak{m}$ are known to have the following characterization [19].

Lemma 2.8. (i) *Any two maximal abelian subspaces $\mathfrak{a}$ are isomorphic to one another under some linear transformation in $H^* \simeq \mathrm{Ad}(H)$.*

(ii) *Every maximal abelian subspace $\mathfrak{a}$ is invariant under the Weyl group $W = C(\mathfrak{a})/N(\mathfrak{a})$, which is a finite subgroup contained in the linear isotropy group H^*, where $C(\mathfrak{a})$ and $N(\mathfrak{a})$ are respectively the centralizer group and normalizer group of $\mathfrak{a}$ in H^*.*

(iii) *Any maximal abelian subspace $\mathfrak{a}$ is the -1 eigenspace in some Cartan subalgebra of the Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ under the grading $\sigma(\mathfrak{m}) = -\mathfrak{m}, \sigma(\mathfrak{h}) = \mathfrak{h}$, and hence $\mathfrak{a}$ is the centralizer of its elements in $\mathfrak{m}$, $\mathfrak{a} = \mathfrak{c}(\mathfrak{a}) \subset \mathfrak{m}$.*

Henceforth the subspaces $\mathfrak{a} \subset \mathfrak{m}$ will be referred to as *Cartan subspaces* of $\mathfrak{m}$ in $\mathfrak{g}$. The close relationship between the subspaces $\mathfrak{a}$ and the Cartan subalgebras of $\mathfrak{g}$ leads to a more explicit description of the algebraic decomposition of $\mathfrak{g}$ with respect to $\mathrm{ad}(e_X)^2$.

Let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. Recall, a root of $\mathfrak{g}$ is a linear function, α, from any Cartan subalgebra $\mathfrak{g}^0$ of $\mathfrak{g}^{\mathbb{C}}$ into $\mathbb{C}$, such that for all vectors w in $\mathfrak{g}^0$, $\mathrm{ad}(w)$ has an eigenvector with the eigenvalue $\alpha(w)$. Let the set of all roots of $\mathfrak{g}$ be denoted Δ. The following lemma is adapted from results in [19], utilizing the fact that $\mathfrak{c}(e_X)^{\perp}_{\mathfrak{m}}$ is contained in $\mathfrak{a}^{\perp}$ as a consequence of the inclusion $\mathfrak{a} \subseteq \mathfrak{c}(e_X)_{\mathfrak{m}}$.

Lemma 2.9. *Let $\Delta_{\mathfrak{m}}$ be the set of roots α of $\mathfrak{g}$ such that $\alpha(a) \neq 0$ for some a in a Cartan subspace $\mathfrak{a} \subset \mathfrak{m}$. Since $\mathfrak{g}$ is compact, $\alpha(a)$ will be an imaginary function (namely $\alpha : \mathfrak{a} \to \mathrm{i}\mathbb{R}$). Then there exists a basis of vectors $\{y_\alpha\}$ in $\mathfrak{c}(e_X)^{\perp}_{\mathfrak{m}}$ and $\{z_\alpha\}$ in $\mathfrak{c}(e_X)^{\perp}_{\mathfrak{h}}$, indexed by a certain subset of the roots α in $\Delta_{\mathfrak{m}}$, satisfying the Lie bracket relations*

$$[y_\alpha, a]_{\mathfrak{g}} = \mathrm{i}\alpha(a)z_\alpha, \qquad [z_\alpha, a]_{\mathfrak{g}} = -\mathrm{i}\alpha(a)y_\alpha \tag{35}$$

for all vectors a in $\mathfrak{a}$. Hence the eigenvalues of the linear map $\mathrm{ad}(a)^2$ on its eigenspaces in $\mathfrak{c}(e_X)^{\perp}_{\mathfrak{m}}, \mathfrak{c}(e_X)^{\perp}_{\mathfrak{h}}$ are given by $\alpha(a)^2 < 0$ (cf. Proposition 2.6). Note $\mathrm{ad}(a)^2$ will be a multiple of the identity precisely when its eigenvalues $\alpha(a)^2$ are equal.

3. Parallel moving frames and bi-Hamiltonian operators

On a n-dimensional Riemannian manifold M, a frame is a basis of the tangent space $T_x M$ at each point x in M, and a coframe is a dual basis of $T_x^* M$. Let $\{e_a\}$ be an orthonormal coframe and $\{e_a^*\}$ its dual frame, $a = 1, \ldots, n$, namely $g(e_a, e_b) = e_a^* \rfloor e_b = \delta_{ab}$. Covariant derivatives of the frame in any direction X at a point in M are given in terms of the frame connection 1-forms $\omega_a{}^b = -\omega_b{}^a$ by $\nabla_X e_a = -(\omega_a{}^b \rfloor X)e_b$. Here the frame structure group (modulo reflections) is the rotation group $SO(n)$. A moving frame along a curve $\gamma(x)$ in M is the restriction of a frame to $T_\gamma M$, or correspondingly a coframe to $T_\gamma^* M$, describing a cross section of the orthonormal frame bundle over γ. $SO(n)$ gauge transformations allow a moving frame to be adapted to γ such that

$$e_a^* = \begin{cases} e_{\parallel}^*, & a = 1; \\ (e_{\perp}^*)_a, & a = 2, \ldots, n \end{cases} \tag{36}$$

where $e_{\parallel}^* = X$ is the unit tangent vector along γ and each $(e_{\perp}^*)_a$ is a unit normal vector with respect to γ. An adapted frame is preserved by the $SO(n-1)$ group of local rotations in the normal space of γ. This gauge freedom can be used to geometrically adapt the connection matrix

$$(\omega_X)_{ab} := \omega_{ab} \rfloor X = g(e_a^*, \nabla_x e_b^*) \tag{37}$$

to γ such that the derivative of the tangent vector in the frame is normal to γ while the derivative of each normal vector is tangential to γ. Such a moving frame is said to be *parallel* [9]:

$$\nabla_x e_{\parallel}^* \perp e_{\parallel}^*, \qquad \nabla_x (e_{\perp}^*)_a \parallel e_{\parallel}^*. \tag{38}$$

Note the connection matrix ω_X^{ab} of a parallel moving frame is skew-symmetric with only its top row and left column being non-zero, as a consequence of the parallel property

$$g(e_{\parallel}^*, \nabla_x e_{\parallel}^*) = 0, \quad g((e_{\perp}^*)_a, \nabla_x (e_{\perp}^*)_b) = 0. \tag{39}$$

This property is preserved by any rigid (x-independent) $SO(n-1)$ rotation in the normal space of γ, which thereby defines the equivalence group for parallel moving frames.

Any parallel moving frame differs from a classical Frenet frame [17], if $n > 2$, by an x-dependent $SO(n-1)$ rotation acting in the normal space of the curve γ (see [35] for a geometric interpretation of this rotation). It is well known that the components of a Frenet connection matrix along γ are differential invariants of the curve [17]. Consequently, the components of a parallel connection matrix are thus differential covariants of γ in sense that they are invariantly defined up to the covariant action of the equivalence group of rigid $SO(n-1)$ rotations.

There is a purely algebraic characterization of the connection matrix of a parallel moving frame [3]. Fix the standard orthonormal basis of $\mathbb{R}^n$, given by n row vectors $\mathbf{m}^a$ whose ath component is 1, with all other components equal to 0, where $a = 1, \ldots, n$. Similarly, fix the standard basis of the Lie algebra $\mathfrak{so}(n)$ induced from the isomorphism $\mathbb{R}^n \wedge \mathbb{R}^n \simeq \mathfrak{so}(n)$, where a wedge denotes the vector outer product. (Note the natural action of $\mathfrak{so}(n)$ on $\mathbb{R}^n$ is defined here by right multiplication of skew-matrices on row vectors.) This basis consists of $n(n-1)/2$ skew-matrices $\mathbf{h}^i = \mathbf{m}^a \wedge \mathbf{m}^b$ whose only non-zero components are 1 in the ath row and bth column, and -1 in the bth row and ath column, where i is identified with (a, b) such that $a < b$, for $a, b = 1, \ldots, n$.

Now, there are associated to the frame e_a^* and its covariant derivatives the $\mathbb{R}^n$-valued linear coframe $e = e_a \mathbf{m}^a$ and the $\mathfrak{so}(n)$-valued linear connection 1-form $\omega = \omega_{ab} \mathbf{m}^a \wedge \mathbf{m}^b$ satisfying soldering relations [20] analogous to the ones (12), where e represents a linear map from $\mathbb{R}^n$ onto $T_x M$. These relations state that ∇ has no torsion

$$\mathrm{d}e - e \wedge \omega = \mathfrak{T} = 0 \tag{40}$$

and carries curvature

$$\mathrm{d}\omega + [\omega, \omega]_{\mathfrak{so}(n)} = \mathfrak{R} \tag{41}$$

determined by the frame components of the Riemann curvature tensor $[\nabla, \nabla]$ of the metric g on M.

Write $\mathbf{m}_\parallel := \mathbf{m}^1$ when $a = 1$, $\mathbf{m}_\perp^a := \mathbf{m}^a$ when $a = 2, \ldots, n$, so thus $e_\parallel = e_1 \mathbf{m}_\parallel$ and $e_\perp = e_a \mathbf{m}_\perp^a$ describe the tangential and normal parts of the linear coframe $e = e_\parallel + e_\perp$ relative to X, with $e_\perp \rfloor X = 0$ for the tangent vector X and $e_\parallel \rfloor Y = 0$ for all normal vectors Y along γ. Consider the derivatives of the coframe e along γ and decompose the connection matrix $\omega_X := \omega \rfloor X = \omega_X^0 + \omega_X^\perp$ into separate parts given by $\omega_X^0 = (\omega_X)_{ab} \mathbf{m}_\perp^a \wedge \mathbf{m}_\perp^b = 0$ and $\omega_X^\perp = (\omega_X)_{1a} \mathbf{m}_\parallel \wedge \mathbf{m}_\perp^a \neq 0$. Observe that in this decomposition $\mathbf{h}_\perp^i := \mathbf{m}_\perp^a \wedge \mathbf{m}_\perp^b$ is a basis for the Lie subalgebra $\mathfrak{so}(n-1) \simeq \mathbb{R}^{n-1} \wedge \mathbb{R}^{n-1}$ where $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ is the subspace orthogonal to $\mathbf{m}_\parallel$, while $\mathbf{h}_\parallel^i := \mathbf{m}_\parallel \wedge \mathbf{m}_\perp^a$ is a basis for the perp space of $\mathfrak{so}(n-1) \subset \mathfrak{so}(n)$.

Proposition 3.1. *A linear coframe $e = e_\parallel + e_\perp$ along a curve $\gamma(x)$ in a Riemannian manifold M is a parallel moving frame iff its connection matrix ω_X vanishes on the Lie subalgebra $\mathfrak{so}(n-1) \subset \mathfrak{so}(n)$ of the $SO(n-1)$ rotation subgroup that leaves the tangent covector $e_\parallel$ invariant under the $SO(n)$ frame structure group.*

This algebraic characterization of the connection matrix of a moving parallel frame for curves in Riemannian geometry has a direct generalization to the setting of Klein geometry following the ideas in [3].

3.1. Construction of parallel moving frames in Klein geometry

Let $\gamma(x)$ be a smooth curve in some n-dimensional compact Riemannian symmetric space $M = G/H$, and write $X = \gamma_x$ for its tangent vector with an affine parametrization so that x is the arclength. Locally on M, introduce a $\mathfrak{m}$-valued linear coframe e and a $\mathfrak{h}$-valued linear connection ω satisfying the soldering relations (12) in terms of the Riemannian metric g and connection ∇ on the manifold M, where $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ is the vector space quotient of the Lie algebras $\mathfrak{g}, \mathfrak{h}$. This structure naturally arises from the Klein geometry of the Lie group G viewed as a principle bundle over M whose structure group is the Lie subgroup $H \subset G$. The projections of e and ω along the curve $\gamma(x)$

$$e_X := e \rfloor X, \qquad \omega_X := \omega \rfloor X \tag{42}$$

provide a moving frame formulation of this curve such that the frame structure group is the linear isotropy group $H^* \simeq \mathrm{Ad}(H)$ acting as local gauge transformations (7). The arclength property of x in this formulation is expressed by $|X|_g = |e_X|_\mathfrak{m} = 1$ so e_X has unit norm in the Cartan–Killing metric on $\mathfrak{m}$ (where $|e_X|_\mathfrak{m}^2 = -\langle e_X, e_X \rangle_\mathfrak{m}$). Informally, e can be regarded as a nondegenerate cross section of the Klein bundle $(T^*M \otimes \mathfrak{m}, H)$ over $\gamma(x)$, where $\mathfrak{m} \simeq \mathbb{R}^n$. This bundle is a reduction [20] of the orthonormal frame bundle of M, with the structure group $SO(n)$

reduced to H. (In fact, $H \subseteq SO(n)$ with equality holding only when $M = G/H$ is a constant curvature manifold, $M = S^n \simeq SO(n+1)/SO(n)$ in the compact case.)

Recall [19], the vector space $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ has the algebraic structure of a union of orbits generated by the group H^* acting on any fixed Cartan space (a maximal abelian subspace) $\mathfrak{a} \subset \mathfrak{m}$ (all of which are isomorphic). Every such subspace $\mathfrak{a}$, moreover, is a finite union of closed convex sets $\mathfrak{a}_*(\mathfrak{m}) \subseteq \mathfrak{a}$ that are mutually isomorphic under the Weyl group $W \subset H^*$ leaving $\mathfrak{a}$ invariant. (Up to closure, these sets $\mathfrak{a}_*$ in $\mathfrak{a}$ each can be identified with Weyl chambers [19] associated to the set of positive roots of the Lie algebra $\mathfrak{g}$ relative to $\mathfrak{m}$.)

Since H^* acts as a group of local gauge transformations on e and ω, a gauge equivalent coframe exists such that, at every point x along γ, e_X is a constant unit vector that lies in the (fixed) Cartan space $\mathfrak{a} \subset \mathfrak{m}$ on which the Weyl group acts as a discrete gauge equivalence group. This leads to the following algebraic characterization for curves $\gamma(x)$.

Proposition 3.2. *The constant unit vector $e_X \subset \mathfrak{a}$ defines an algebraic invariant of the curve γ. Thus, all curves on $M = G/H$ can be divided into algebraic classes in one-to-one correspondence with the unit-norm elements in the closed convex set $\mathfrak{a}_*(\mathfrak{m}) \simeq \mathfrak{a}/W$ as determined by e_X.*

For a given vector e_X, a linear coframe e along γ is determined only up to local gauge transformations (7) given by the group $H_X^* \subset H^*$ that preserves $e_X \subset \mathfrak{a}$. Such transformations will be referred to as the *isotropy gauge group* of γ. The isotropy gauge freedom can be used to adapt e and ω to γ in a preferred way, generalizing the algebraic notion of a parallel moving frame for curves in Riemannian geometry stated in Proposition 3.1.

Definition 3.3. A $\mathfrak{m}$-valued linear coframe e along a curve $\gamma(x)$ in a Riemannian symmetric space $M = G/H$ is said to be *H-parallel* if its associated $\mathfrak{h}$-valued linear connection ω_X in the tangent direction $X = \gamma_x$ of the curve lies in the perp space $\mathfrak{c}(e_X)_{\mathfrak{h}}^{\perp}$ of the Lie subalgebra $\mathfrak{c}(e_X)_{\mathfrak{h}} \subset \mathfrak{h}$ of the isotropy gauge group $H_X^* \subset H^*$ that preserves e_X given by a fixed unit vector in $\mathfrak{a} \subset \mathfrak{m}$.

To explain the geometrical meaning of such a coframe, first decompose the dual frame $e^* = e_{\parallel}^* + e_{\perp}^*$ and the connection $\omega = \omega^{\parallel} + \omega^{\perp}$ according to

$$\mathfrak{m} = \mathfrak{c}(e_X)_{\mathfrak{m}} \oplus \mathfrak{c}(e_X)_{\mathfrak{m}}^{\perp}, \qquad \mathfrak{h} = \mathfrak{c}(e_X)_{\mathfrak{h}} \oplus \mathfrak{c}(e_X)_{\mathfrak{h}}^{\perp}, \tag{43}$$

relative to the unit vector $e_X \subset \mathfrak{a} \subset \mathfrak{m}$. The $\mathfrak{m}$-valued linear frame e^* will provide a framing of the curve γ in M (namely a linear map from the vector space $\mathfrak{m}^* \simeq \mathbb{R}^n$ onto $T_x^* M$ over $\gamma(x)$) once a basis expansion for $\mathfrak{m}, \mathfrak{h}$ is introduced. Note the condition for e^* to be *H-parallel* is

$$\omega_X^{\parallel} := \omega^{\parallel} \lrcorner X = 0 \tag{44}$$

along with

$$D_x e_X = 0, \qquad |e_X|_{\mathfrak{m}} = 1. \tag{45}$$

Remark. This property $\omega_X^{\parallel} = 0$ is maintained by rigid (x-independent) gauge transformations in H_X^* that preserve e_X, since under these transformations, $\mathrm{Ad}(h^{-1})X = X$ implies $\mathrm{Ad}(h^{-1})\omega_X^{\perp}$ belongs to $\mathfrak{c}(e_X)_{\mathfrak{h}}$ and hence the transformed linear connection (7) satisfies $\tilde{\omega}_X^{\parallel} = 0$. All such transformations comprise the *equivalence group*, denoted $H_{\parallel}^*$, for *H-parallel* moving frames. The infinitesimal equivalence group is simply given by the centralizer Lie subalgebra $\mathfrak{c}(e_X)_{\mathfrak{h}} \subset \mathfrak{h}$, as seen from the Lie bracket relations (30) and (31). Note $H_{\parallel}^*$ preserves the decomposition (43) of the tangent space $T_\gamma M \simeq_G = \mathfrak{m}$ into algebraically parallel and perpendicular parts with respect to X.

Let $\{\mathbf{m}_{\parallel}^a, \mathbf{h}_{\parallel}^i\}$ be respective orthonormal bases for the subspaces $\mathfrak{c}(e_X)_{\mathfrak{m}}, \mathfrak{c}(e_X)_{\mathfrak{h}}$, so

$$\mathrm{ad}(e_X)\mathbf{m}_{\parallel}^a = \mathrm{ad}(e_X)\mathbf{h}_{\parallel}^i = 0 \tag{46}$$

(where $a = 1, \ldots, \dim \mathfrak{c}(e_X)_{\mathfrak{m}}$; $i = 1, \ldots, \dim \mathfrak{c}(e_X)_{\mathfrak{h}}$). Recall, the perp subspaces $\mathfrak{c}(e_X)_{\mathfrak{m}}^{\perp}, \mathfrak{c}(e_X)_{\mathfrak{h}}^{\perp}$ are isomorphic under the linear map $\mathrm{ad}(e_X)$, and so let $\{\mathbf{m}_{\perp}^a, \mathbf{h}_{\perp}^i\}$ be respective orthonormal bases for $\mathfrak{c}(e_X)_{\mathfrak{m}}^{\perp} \simeq \mathfrak{c}(e_X)_{\mathfrak{h}}^{\perp}$ related by

$\mathbf{m}_\perp^a = \mathrm{ad}(e_X)^{-1} \mathbf{h}_\perp^i$ (where $a, i = 1, \ldots, \dim \mathfrak{c}(e_X)_{\mathfrak{m}}^\perp = \dim \mathfrak{c}(e_X)_{\mathfrak{h}}^\perp$). Orthonormality of each basis is given by

$$\langle \mathbf{m}_\parallel^a, \mathbf{m}_\parallel^b \rangle_{\mathfrak{m}} = \langle \mathbf{m}_\perp^a, \mathbf{m}_\perp^b \rangle_{\mathfrak{m}} = -\delta^{ab}, \tag{47}$$

$$\langle \mathbf{h}_\parallel^i, \mathbf{h}_\parallel^j \rangle_{\mathfrak{h}} = \langle \mathbf{h}_\perp^i, \mathbf{h}_\perp^j \rangle_{\mathfrak{h}} = -\delta^{ij}, \tag{48}$$

and

$$\langle \mathbf{m}_\perp^a, \mathbf{m}_\parallel^b \rangle_{\mathfrak{m}} = \langle \mathbf{h}_\perp^i, \mathbf{h}_\parallel^j \rangle_{\mathfrak{h}} = 0. \tag{49}$$

Now, write

$$(e_\parallel^*)_a = -\langle \mathbf{m}_\parallel^a, e^* \rangle_{\mathfrak{m}}, \qquad (e_\perp^*)_a = -\langle \mathbf{m}_\perp^a, e^* \rangle_{\mathfrak{m}}, \tag{50}$$

and

$$(\omega^\parallel)_i = -\langle \mathbf{h}_\parallel^i, \omega \rangle_{\mathfrak{h}}, \qquad (\omega^\perp)_i = -\langle \mathbf{h}_\perp^i, \omega \rangle_{\mathfrak{h}}, \tag{51}$$

yielding the basis expansion of e^*, ω. As a consequence of properties (46)–(49), the vectors $(e_\parallel^*)_a$ and $(e_\perp^*)_a$ will be said, respectively, to be *algebraically parallel* ($\parallel$) and *perpendicular* ($\perp$) to the tangent vector $X = \gamma_x$ along the curve. This framing is adapted to γ if it contains X as one of the vectors parallel to X, (say for $a = 1$)

$$(e_\parallel^*)_1 = X, \tag{52}$$

which can be achieved by letting $\mathbf{m}_\parallel^1 = e_X$ be one of the basis vectors of $\mathfrak{c}(e_X)_{\mathfrak{m}}$.

Lemma 3.4. *The frame vectors $\{(e_\parallel^*)_a, (e_\perp^*)_a\}$ provide an orthonormal basis of $T_\gamma M$ over the curve $\gamma(x)$. Through the soldering formula (25), their covariant derivatives in the tangent direction $X = \gamma_x$ are determined by the connection matrices $(\omega_X^\perp)_i c^{ib}{}_a$ with $(\omega_X^\parallel)_i c^{ib}{}_a = 0$.*

The Lie bracket relation (31) between $\mathfrak{c}(e_X)_{\mathfrak{h}}^\perp$ and $\mathfrak{c}(e_X)_{\mathfrak{m}}$ now leads to an explicit geometric interpretation of the H-parallel property. Note Y is algebraically $\parallel$ to X iff $\mathrm{ad}(e_X)e_Y = [e_X, e_Y] = 0$ namely $e_Y := e \rfloor Y$ belongs to $\mathfrak{c}(e_X)_{\mathfrak{m}}$. Likewise Y is algebraically $\perp$ to X iff e_Y belongs to $\mathfrak{c}(e_X)_{\mathfrak{m}}^\perp$.

Theorem 3.5. *For any H-parallel frame along a curve $\gamma(x)$, the covariant derivatives $\nabla_x(e_\parallel^*)_a$, $a = 1, \ldots, \dim \mathfrak{c}(e_X)_{\mathfrak{m}}$, are algebraically $\perp$ to the tangent vector $X = \gamma_x$. The other covariant derivatives $\nabla_x(e_\perp^*)_a$, $a = 1, \ldots, \dim \mathfrak{c}(e_X)_{\mathfrak{m}}^\perp$, are algebraically $\parallel$ to X iff the Lie bracket relation $[\mathfrak{c}(e_X)_{\mathfrak{m}}^\perp, \mathfrak{c}(e_X)_{\mathfrak{h}}^\perp]_{\mathfrak{g}} \subseteq \mathfrak{c}(e_X)_{\mathfrak{m}}$ holds, which is always the case whenever the linear map $\mathrm{ad}(e_X)^2$ is a multiple of the identity on the vector spaces $\mathfrak{c}(e_X)_{\mathfrak{m}}^\perp \simeq \mathfrak{c}(e_X)_{\mathfrak{h}}^\perp$ (cf. Corollary 2.4).*

These geometric properties of an H-parallel frame are a strict generalization of the Riemannian case (38) whenever $\dim \mathfrak{c}(e_X)_{\mathfrak{m}}^\perp > 1$, or $[\mathfrak{c}(e_X)_{\mathfrak{m}}^\perp, \mathfrak{c}(e_X)_{\mathfrak{h}}^\perp]_{\mathfrak{g}} \cap \mathfrak{c}(e_X)_{\mathfrak{m}}^\perp \neq 0$.

There is a further geometric meaning for an H-parallel connection itself. Recall, the principal normal along any affine-parametrized curve is the normal vector [17]

$$N = \nabla_x \gamma_x. \tag{53}$$

Proposition 3.6. *In an H-parallel adapted frame along a curve $\gamma(x)$, the principal normal is given by $N = \langle \mathrm{ad}(e_X)\omega_X^\perp, e^* \rangle_{\mathfrak{m}}$ with the frame components $N^a = e_a \rfloor N = (\omega_X^\perp)_i c^{i1}{}_a := (\omega_X^\perp)_a$ where $c^{i1}{}_a = \mathrm{ad}(e_X)_a{}^i$ is an invertible matrix ($a, i = 1, \ldots, \dim \mathfrak{c}(e_X)_{\mathfrak{m}}^\perp = \dim \mathfrak{c}(e_X)_{\mathfrak{h}}^\perp$). Moreover, these components are differential covariants of the curve with respect to the rigid equivalence group (of x-independent linear transformations) $H_\parallel^* \subset H^*$ (preserving the framing). Namely the set of components $(\omega_X^\perp)_i c^{i1}{}_a$ is invariantly defined by $\gamma(x)$ up to the covariant action of the group $H_\parallel^*$.*

An important final remark now is that the H-parallel condition (44) on the linear connection ω can always be achieved by means of a suitable gauge transformation (7) as follows. Under the isotropy gauge group H_X^*, $\omega_X^\parallel$ is

transformed to $\tilde{\omega}_X^{\parallel} = \mathrm{Ad}(h^{-1})\omega_X^{\parallel} + h^{-1}D_x h$ where h is an arbitrary smooth function from the curve $\gamma(x) \subset M$ into the Lie subgroup of $H_X^* \subset H$, satisfying $\mathrm{Ad}(h^{-1})e_X = 0$. Condition (44) applied to $\tilde{\omega}_X^{\parallel}$ yields an ODE on $h(x)$, $h^{-1}D_x h = -\mathrm{Ad}(h^{-1})\omega_X^{\parallel}$, for which local existence of a solution is a standard result. (In particular, in the adjoint representation of H and $\mathfrak{h}$, this ODE becomes a linear homogeneous matrix differential equation $D_x\mathrm{Ad}(h) = -\mathrm{Ad}(h)\mathrm{ad}(\omega_X^{\parallel})$.) This result is a straightforward generalization of the standard existence of a parallel moving frame in the Riemannian setting.

Proposition 3.7. *By means of a gauge transformation in the isotropy gauge group H_X^*, an H-parallel frame $\{(e_{\parallel}^*)_a, (e_{\perp}^*)_a\}$ exists for any curve $\gamma(x)$ in a given Riemannian symmetric space G/H and is unique up to the equivalence group $H_{\parallel}^* \subset H^*$ consisting of rigid (x-independent) gauge transformations that preserve the algebraic structure (43)–(45) of the framing along $\gamma(x)$.*

4. Derivation of bi-Hamiltonian operators from non-stretching curve flows

Consider an arbitrary smooth flow $\gamma(t, x)$ of any smooth curve in an n-dimensional Riemannian symmetric space $M = G/H$. Write $X = \gamma_x$ for the tangent vector and $Y = \gamma_t$ for the evolution vector along the curve. Provided the flow is transverse to the curve, namely X and Y are not parallel vectors, then $\gamma(t, x)$ will describe a smooth two-dimensional surface immersed in the manifold M.

The torsion and curvature equations of the Riemannian connection ∇ on M restricted to this surface $\gamma(t, x)$ are given by

$$T(X, Y) = \nabla_x \gamma_t - \nabla_t \gamma_x = 0, \tag{54}$$

$$R(X, Y) = [\nabla_x, \nabla_t] = -\mathrm{ad}_{\mathfrak{m}}([\gamma_x, \gamma_t]_{\mathfrak{m}}), \tag{55}$$

where $[\cdot, \cdot]_{\mathfrak{m}}$ is the Lie bracket on $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ with the canonical identification $\mathfrak{m} \simeq_G T_x M$. There is a straightforward frame formulation for these equations obtained from the pullback of the Cartan structure equations (10) for any $\mathfrak{m}$-valued linear coframe e and $\mathfrak{h}$-valued linear connection ω on the surface $\gamma(t, x)$:

$$D_x e_t - D_t e_x + [\omega_x, e_t]_{\mathfrak{g}} - [\omega_t, e_x]_{\mathfrak{g}} = 0, \tag{56}$$

$$D_x \omega_t - D_t \omega_x + [\omega_x, \omega_t]_{\mathfrak{h}} = -[e_x, e_t]_{\mathfrak{m}}, \tag{57}$$

where

$$e_x := e \rfloor X = e \rfloor \gamma_x, \quad e_t := e \rfloor Y = e \rfloor \gamma_t, \tag{58}$$

$$\omega_x := \omega \rfloor X = \omega \rfloor \gamma_x, \quad \omega_t := \omega \rfloor Y = \omega \rfloor \gamma_t. \tag{59}$$

These structure equations coincide with the zero-curvature equation (4) for G viewed as a H-bundle in a local trivialization over the surface $\gamma(t, x)$ in M. In particular, in local coordinates for $G \approx U \times H$, the $\mathfrak{g}$-valued connection 1-form $e + \omega = {}^{\mathfrak{g}}\omega$ coincides with the pullback of the left-invariant Maurer–Cartan form on G to $U|_{\gamma} \subset M$, locally giving a soldering of the frame geometry of G/H onto the Riemannian geometry of the manifold M.

4.1. Non-stretching geometric curve flows

A curve flow $\gamma(t, x)$ is *non-stretching* (inextensible) if it preserves the arclength $\mathrm{d}s = |\gamma_x|\mathrm{d}x$ at every point on the curve, namely $D_t|\gamma_x| = 0$ or equivalently $\nabla_t \gamma_x (= \nabla_x \gamma_t)$ is orthogonal to γ_x in the Riemannian metric g on M. Now let $\gamma(t, x)$ be a non-stretching curve flow given in terms of an affine parametrization $|\gamma_x| = 1$, without loss of generality, where x measures arclength along the curve. For any such flow, there exists an H-parallel frame formulation obtained through Proposition 3.7. In particular, a suitable local gauge transformation in $H^* \simeq \mathrm{Ad}(H)$ (given by a change of coordinates for the local trivialization of G) can be applied to the linear coframe e and linear connection ω to produce an H-parallel moving frame along each curve in the flow $\gamma(t, x)$, depending smoothly on t.

Lemma 4.1. *In an H-parallel frame formulation for non-stretching curve flows, the projections (58) and (59) of e and ω in the tangent direction $X = \gamma_x$ and the flow direction $Y = \gamma_t$ have the following properties:*

(i) e_x belongs to some Cartan space $\mathfrak{a} \subset \mathfrak{m}$, is constant ($D_x e_x = 0$), and has the normalization $|e_x|_\mathfrak{m} = 1$;

(ii) $e_t := h_\parallel + h_\perp$ and $\omega_t := \varpi^\parallel + \varpi^\perp$ each decompose into algebraically parallel and perpendicular parts relative to e_x, namely $h_\parallel$ and $\varpi^\parallel$ belong to the centralizer subspaces $\mathfrak{m}_\parallel := \mathfrak{c}(e_X)_\mathfrak{m}$ and $\mathfrak{h}_\parallel := \mathfrak{c}(e_X)_\mathfrak{h}$ while $h_\perp$ and $\varpi^\perp$ belong to the perp spaces $\mathfrak{m}_\perp := \mathfrak{c}(e_X)_\mathfrak{m}^\perp$ and $\mathfrak{h}_\perp := \mathfrak{c}(e_X)_\mathfrak{h}^\perp$;

(iii) $u := \omega_x$ belongs to the perp space $\mathfrak{h}_\perp$;

(iv) $h_\perp, e_x, u$ respectively determine the perpendicular flow vector $Y_\perp = (\gamma_t)_\perp = -\langle h_\perp, e^* \rangle_\mathfrak{m}$, the tangent vector $X = \gamma_x = -\langle e_x, e^* \rangle_\mathfrak{m}$, and the principal normal vector $N = \nabla_x \gamma_x = -\langle \mathrm{ad}(e_x)u, e^* \rangle_\mathfrak{m}$ for the curve.

The Lie bracket relations in Proposition 2.5 on the spaces $\mathfrak{m} = \mathfrak{m}_\parallel \oplus \mathfrak{m}_\perp$ and $\mathfrak{h} = \mathfrak{h}_\parallel \oplus \mathfrak{h}_\perp$ lead to a corresponding decomposition of the Cartan structure equations (56) and (57) expressed in terms of the variables $e_x, h_\parallel, h_\perp, \varpi^\parallel, \varpi^\perp, u$. The torsion equation becomes

$$0 = D_x h_\parallel + [u, h_\perp]_\parallel, \tag{60}$$

$$0 = \mathrm{ad}(e_x)\varpi^\perp + D_x h_\perp + [u, h_\parallel] + [u, h_\perp]_\perp, \tag{61}$$

while the curvature equation becomes

$$0 = D_x \varpi^\parallel + [u, \varpi^\perp]_\parallel, \tag{62}$$

$$\mathrm{ad}(e_x)h_\perp = D_t u - D_x \varpi^\perp + [\varpi^\parallel, u] + [\varpi^\perp, u]_\perp, \tag{63}$$

where $[\cdot, \cdot]_\parallel$ denotes the restriction of the Lie bracket on $\mathfrak{g}$ to $\mathfrak{m}_\parallel$ or $\mathfrak{h}_\parallel$, and likewise $[\cdot, \cdot]_\perp$ denotes the restriction to $\mathfrak{m}_\perp$ or $\mathfrak{h}_\perp$. Note that, in the present notation, the Lie bracket relations consist of

$$[\mathfrak{m}_\parallel, \mathfrak{m}_\parallel] \subseteq \mathfrak{h}_\parallel, \qquad [\mathfrak{h}_\parallel, \mathfrak{m}_\parallel] \subseteq \mathfrak{m}_\parallel, \qquad [\mathfrak{h}_\parallel, \mathfrak{h}_\parallel] \subseteq \mathfrak{h}_\parallel, \tag{64}$$

$$[\mathfrak{m}_\parallel, \mathfrak{m}_\perp] \subseteq \mathfrak{h}_\perp, \qquad [\mathfrak{h}_\parallel, \mathfrak{m}_\perp] \subseteq \mathfrak{m}_\perp, \qquad [\mathfrak{h}_\perp, \mathfrak{m}_\parallel] \subseteq \mathfrak{m}_\perp, \qquad [\mathfrak{h}_\parallel, \mathfrak{h}_\perp] \subseteq \mathfrak{h}_\perp, \tag{65}$$

so thus

$$[\mathfrak{m}_\parallel, \mathfrak{m}_\parallel]_\perp = [\mathfrak{h}_\parallel, \mathfrak{m}_\parallel]_\perp = [\mathfrak{h}_\parallel, \mathfrak{h}_\parallel]_\perp = 0, \tag{66}$$

and

$$[\mathfrak{m}_\parallel, \mathfrak{m}_\perp]_\parallel = [\mathfrak{h}_\parallel, \mathfrak{m}_\perp]_\parallel = [\mathfrak{h}_\perp, \mathfrak{m}_\parallel]_\parallel = [\mathfrak{h}_\parallel, \mathfrak{h}_\perp]_\parallel = 0. \tag{67}$$

Therefore the only brackets with a possibly nontrivial decomposition are $[\mathfrak{m}_\perp, \mathfrak{m}_\perp]$ and $[\mathfrak{h}_\perp, \mathfrak{h}_\perp]$.

Lemma 4.2. *The Cartan structure equations for any H-parallel linear coframe e and linear connection ω pulled back to the two-dimensional surface $\gamma(t, x)$ determine a flow on $u = \omega_x$ given by*

$$u_t = D_x \varpi^\perp + [u, \varpi^\parallel] + [u, \varpi^\perp]_\perp + \mathrm{ad}(e_x)h_\perp, \tag{68}$$

$$\varpi^\perp = -\mathrm{ad}(e_x)^{-1}(D_x h_\perp + [u, h_\parallel] + [u, h_\perp]_\perp), \tag{69}$$

with

$$h_\parallel = -D_x^{-1}[u, h_\perp]_\parallel, \quad \varpi^\parallel = -D_x^{-1}[u, \varpi^\perp]_\parallel, \tag{70}$$

where D_x^{-1} denotes the formal inverse of the total x-derivative operator D_x.

Note that, once $h_\perp$ is specified, this determines both the flow on u and the curve flow of γ itself, where $h_\parallel, \varpi^\parallel, \varpi^\perp$ are given in terms of $h_\perp$ by Eqs. (69) and (70). A natural class of geometric flows will now be introduced.

Definition 4.3. A curve flow is said to be *geometric* if $h_\perp$ is an equivariant $\mathfrak{m}_\perp$-valued function of $(x, u, u_x, \ldots)$ under the action of the rigid (x-independent) group of linear transformations $H_\parallel^*$, where the arclength variable x is an invariant of the curve, and the components of flow variable u are differential covariants of the curve (cf Proposition 3.6).

In the case of polynomial geometric flows, $\mathfrak{m}_\perp$-equivariance means that the function $h_\perp$ is constructible in terms of $(u, u_x, \ldots)$ together with e_x and x using just the Lie bracket and Cartan–Killing inner product on these variables. Such flows correspond to curve flows $\gamma(t, x)$ such that $\gamma_t = f(x, \gamma_x, \gamma_{xx}, \ldots)$ is constructed using the metric $g(\cdot, \cdot)$ and the linear map $\mathrm{ad}_x(\cdot)^2$ on $T_x M \simeq_G \mathfrak{m}$ via the relations (i)–(iv) in Lemma 4.1.

Proposition 4.4. *For a given polynomial geometric flow on u, the curve γ obeys a G-invariant flow equation in M determined through the identifications*

$$\gamma_x \leftrightarrow e_x, \quad \nabla_x \leftrightarrow D_x + [u, \cdot]_{\mathfrak{m}} := \mathcal{D}_x, \quad \mathrm{ad}_x(\cdot) \leftrightarrow \mathrm{ad}_{\mathfrak{m}}(\cdot), \quad g(\cdot, \cdot) \leftrightarrow -\langle \cdot, \cdot \rangle_{\mathfrak{m}}. \tag{71}$$

4.2. Hamiltonian operators and bi-Hamiltonian structure

As a main result, the flow equation on u will now be shown to possess an elegant bi-Hamiltonian structure. To begin, recall that $\mathrm{ad}(e_x)$ as a linear map gives an isomorphism between the spaces $\mathfrak{m}_\perp$ and $\mathfrak{h}_\perp$. Write $h^\perp := \mathrm{ad}(e_x)h_\perp$ for the image of $h_\perp$ which belongs to the same perp space $\mathfrak{h}_\perp$ as u. Expressing u in terms of $h^\perp$ and $\varpi^\perp$ then leads to the following operator form for the flow equation (68):

$$u_t = \mathcal{H}(\varpi^\perp) + h^\perp, \quad \varpi^\perp = \mathcal{J}(h^\perp), \tag{72}$$

where

$$\mathcal{H} = \mathcal{K}|_{\mathfrak{h}_\perp}, \qquad \mathcal{J} = -\mathrm{ad}(e_x)^{-1}\mathcal{K}|_{\mathfrak{m}_\perp}\mathrm{ad}(e_x)^{-1} \tag{73}$$

are linear operators which act on $\mathfrak{h}_\perp$-valued functions and are invariant under $H_\parallel^*$, as defined in terms of the operator

$$\mathcal{K} := D_x + [u, \cdot]_\perp - [u, D_x^{-1}[u, \cdot]_\parallel]. \tag{74}$$

Theorem 4.5. *$\mathcal{H}, \mathcal{J}$ are compatible Hamiltonian cosymplectic and symplectic operators with respect to the $\mathfrak{h}_\perp$-valued flow variable u. Consequently, for geometric flows in terms of $h^\perp(x, u, u_x, \ldots)$ the flow equation has an $H_\parallel^*$-invariant Hamiltonian form*

$$u_t = \mathcal{H}_0(\delta\mathfrak{H}/\delta u) \tag{75}$$

where $\mathcal{H}_0 = \mathcal{H} + \mathcal{J}^{-1}$ is a Hamiltonian operator, such that

$$\delta\mathfrak{H}/\delta u = \varpi^\perp = \mathcal{J}(h^\perp) \tag{76}$$

for some Hamiltonian functional $\mathfrak{H} = \int H(x, u, u_x, \ldots)\mathrm{d}x$. Moreover, all such Hamiltonian flows have a (compatible) second Hamiltonian structure

$$u_t = \mathcal{J}_0(\delta\mathfrak{F}/\delta u) \tag{77}$$

given by $\mathcal{J}_0 = \mathcal{R}^{-1}\mathcal{H}_0$, for a Hamiltonian functional $\mathfrak{F} = \int F(x, u, u_x, \ldots)\mathrm{d}x$ satisfying

$$\delta\mathfrak{F}/\delta u = \mathcal{R}(\varpi^\perp) = \mathcal{J}(\mathcal{R}(h^\perp)) \tag{78}$$

where $\mathcal{R} = \mathcal{H}\mathcal{J}$ is a (hereditary) recursion operator.

Here $\delta\mathfrak{H}/\delta u$ denotes the variational derivative of the real-valued functional $\mathfrak{H}$. More precisely, $\delta\mathfrak{H}/\delta u$ is defined as the unique $\mathfrak{h}_\perp$-valued function of $(x, u, u_x, \ldots)$ such that

$$-\int \langle \delta\mathfrak{H}/\delta u, h^\perp \rangle \mathrm{d}x = \delta_{h^\perp}\mathfrak{H} \tag{79}$$

formally holds for all $\mathfrak{h}_\perp$-valued functions $h^\perp$, where $\delta_{h^\perp} = \mathrm{pr}(h^\perp \cdot \partial/\partial u)$ denotes the variation induced by $h^\perp$, namely the prolongation of the given vector field $h^\perp \cdot \partial/\partial u$ on J^∞, and where $\langle \cdot, \cdot \rangle$ denotes the (negative-definite) Cartan–Killing inner product on $\mathfrak{h}_\perp$. (A dot will denote summation over Lie algebra components with respect to any fixed basis.)

The definition of cosymplectic and symplectic operators and a proof of this theorem will be given in Sections 4.3–4.5. A summary of the basic theory of bi-Hamiltonian operators and Hamiltonian structures for scalar evolution equations is given in [29,15]. The presentation here will extend this theory to a Lie-algebra-valued setting.

One main property characterizing Hamiltonian operators consists of the symplectic structure they induce on the x-jet space $(x, u, u_x, \ldots) := J^\infty$ of the flow variable u. In particular, there is associated to $\mathcal{H}$ the Poisson bracket defined by

$$\{\mathfrak{H}, \mathfrak{G}\}_{\mathcal{H}} = - \int \langle \mathcal{H}(\delta\mathfrak{G}/\delta u), \delta\mathfrak{H}/\delta u \rangle \mathrm{d}x \tag{80}$$

for any pair of functionals $\mathfrak{H}$, $\mathfrak{G}$. This bracket is skew-symmetric

$$\{\mathfrak{H}, \mathfrak{G}\}_{\mathcal{H}} = -\{\mathfrak{G}, \mathfrak{H}\}_{\mathcal{H}} \tag{81}$$

and obeys the Jacobi identity

$$\{\mathfrak{H}, \{\mathfrak{G}, \mathfrak{F}\}_{\mathcal{H}}\}_{\mathcal{H}} + \{\mathfrak{G}, \{\mathfrak{F}, \mathfrak{H}\}_{\mathcal{H}}\}_{\mathcal{H}} + \{\mathfrak{F}, \{\mathfrak{H}, \mathfrak{G}\}_{\mathcal{H}}\}_{\mathcal{H}} = 0 \tag{82}$$

as a consequence of $\mathcal{H}$ being cosymplectic. Let $h^\perp(x, u, u_x, \ldots)$ be a function with values in the space $\mathfrak{h}_\perp$. Then, on the x-jet space J^∞, $h^\perp \cdot \partial/\partial u$ is a *Hamiltonian vector field* with respect to $\mathcal{H}$ if there exists a (Hamiltonian) functional $\mathfrak{H}$ such that

$$\delta_{h^\perp}\mathfrak{G} = \{\mathfrak{G}, \mathfrak{H}\}_{\mathcal{H}} \tag{83}$$

for all functionals $\mathfrak{G}$. Associated to $\mathcal{J}$ is a symplectic 2-form defined by

$$\boldsymbol{\omega}(h_1^\perp \cdot \partial/\partial u, h_2^\perp \cdot \partial/\partial u)_{\mathcal{J}} = - \int \langle h_1^\perp, \mathcal{J}(h_2^\perp) \rangle \mathrm{d}x \tag{84}$$

for any pair of Hamiltonian vector fields given by $h_1^\perp$, $h_2^\perp$. The property that $\boldsymbol{\omega}$ is symplectic corresponds to the skew-symmetry and Jacobi identity of the Poisson bracket $\{\cdot, \cdot\}_{\mathcal{J}^{-1}}$ defined using the formal inverse of $\mathcal{J}$. (Consequently, $\mathcal{J}^{-1}$ is formally cosymplectic, and conversely $\mathcal{H}^{-1}$ is formally symplectic.) This 2-form $\boldsymbol{\omega}$ gives a canonical pairing between Hamiltonian vector fields and corresponding covector fields defined as follows. Let $\varpi^\perp(x, u, u_x, \ldots)$ be a function with values in the space $\mathfrak{h}_\perp$. Then the covector field $\varpi^\perp \cdot \mathrm{d}u$ is dual to a Hamiltonian vector field $h^\perp \cdot \partial/\partial u$ if

$$\varpi^\perp \cdot \mathrm{d}u := \boldsymbol{\omega}(\cdot, h^\perp \cdot \partial/\partial u)_{\mathcal{J}} \tag{85}$$

holds with respect to $\mathcal{J}$. As a consequence of $\mathcal{J}$ being symplectic, such covector fields are *variational*, namely there exists a (Hamiltonian) functional $\mathfrak{E}$ such that

$$\delta_{h^\perp}\mathfrak{E} = - \int \langle \varpi^\perp, h^\perp \rangle \mathrm{d}x \tag{86}$$

for all vector fields $h^\perp \cdot \partial/\partial u$.

Proposition 4.6. *The Hamiltonian (cosymplectic and symplectic) operators* (73) *give mappings* $h^\perp \cdot \partial/\partial u \to \mathcal{J}(h^\perp) \cdot \mathrm{d}u$ *and* $\varpi^\perp \cdot \mathrm{d}u \to \mathcal{H}(\varpi^\perp) \cdot \partial/\partial u$ *between Hamiltonian vector fields and variational covector fields on the x-jet space of the flow variable u.*

Theorem 4.5 and Proposition 4.6 provide a broad generalization of known results on the geometric origin of bi-Hamiltonian structures in Riemannian symmetric spaces $M = G/H$. Special cases were first derived in [30, 1] for constant curvature spaces $SO(N + 1)/SO(N) \simeq S^N$ and subsequently generalized in [3] to the spaces $G/SO(N)$, covering all examples in which the bi-Hamiltonian structure is a $O(N - 1)$-invariant (namely, $H_\parallel^* = O(N - 1) \subset SO(N)$). The proof of Theorem 4.5 in the general case is similar to the special case proven in [30] where $\mathfrak{h}_\parallel = \mathfrak{so}(N - 1) \simeq \mathbb{R}^{N-1} \wedge \mathbb{R}^{N-1}$, $\mathfrak{h}_\perp \simeq \mathbb{R}^{N-1}$ such that $[\mathfrak{h}_\perp, \mathfrak{h}_\parallel] \subseteq \mathfrak{h}_\perp$ and $[\mathfrak{h}_\perp, \mathfrak{h}_\perp] = \mathfrak{h}_\parallel$. Note these Lie bracket relations correspond to the geometrical properties (39) of a Riemannian parallel frame [3]. There are two main differences in the general case. Firstly, the operators $\mathcal{H}$, $\mathcal{J}$ have an additional term arising from the Lie bracket

structure $[\mathfrak{h}_\perp, \mathfrak{h}_\perp] \cap \mathfrak{h}_\perp \neq 0$ as permitted by the more general algebraic notion of a parallel frame in Definition 3.3 and Theorem 3.5. Secondly, these operators are formulated here in a manifestly $H_\parallel^*$-invariant fashion, using just the intrinsic Lie bracket on $\mathfrak{h} = \mathfrak{h}_\parallel \oplus \mathfrak{h}_\perp$, as opposed to a more cumbersome $\mathbb{R}^{N-1}$-component form used in [30].

4.3. Proof of cosymplectic property for $\mathcal{H}$

A linear operator $\mathcal{H}$ that acts on $\mathfrak{h}_\perp$-valued functions in the x-jet space J^∞ of the flow variable u is a *Hamiltonian cosymplectic operator* [29,15] iff its associated Poison bracket (80) is skew-symmetric and obeys the Jacobi identity.

Skew-symmetry (81) is equivalent to $\mathcal{H}$ being skew-adjoint with respect to the natural inner product $\int \langle f_1, f_2 \rangle \mathrm{d}x$ on $\mathfrak{h}_\perp$-valued functions f on J^∞ induced by the Cartan–Killing inner product on $\mathfrak{h}_\perp$. This condition can be formulated in a pointwise manner on J^∞ as follows: $\mathcal{H}$ is skew-adjoint iff

$$\langle f, \mathcal{H}(f) \rangle \equiv 0 \text{ modulo a total } x\text{-derivative} \tag{87}$$

(namely, $\langle f, \mathcal{H}(f) \rangle = D_x \Upsilon$ for some scalar function Υ).

Proposition 4.7. *The linear operator $\mathcal{H} = D_x + [u, \cdot]_\perp - [u, D_x^{-1}[u, \cdot]_\parallel]$ in Theorem 4.5 is skew-adjoint.*

Proof. The l.h.s. of (87) is given by

$$\langle f, D_x f + [u, f]_\perp - [u, D_x^{-1}[u, f]_\parallel] \rangle \equiv \langle f, [u, f] \rangle - \langle f, [u, D_x^{-1}[u, f]_\parallel] \rangle \tag{88}$$

since $\langle f, D_x f \rangle = -\frac{1}{2} D_x |f|^2$. The remaining two terms can be simplified by means of the identities

$$\langle A, [B, C_\perp] \rangle = \text{cyclic}, \qquad \langle A, [B, C_\parallel] \rangle = \langle C_\parallel, [A, B] \rangle, \tag{89}$$

for functions A, B with values in $\mathfrak{h}_\perp$ and C in $\mathfrak{h}$. Thus

$$\langle f, [u, f] \rangle = -\langle u, [f, f] \rangle = 0 \tag{90}$$

and

$$\langle f, [u, A(f)] \rangle = -\langle A(f), [u, f]_\parallel \rangle = \frac{1}{2} D_x |A(f)|^2 \equiv 0 \tag{91}$$

where $A(f) = D_x^{-1}[u, f]_\parallel$. Hence the l.h.s. of (87) reduces to a total x-derivative, which completes the proof.   □

For a skew-adjoint operator $\mathcal{H}$, the Jacobi property (82) is well known to be equivalent to the vanishing of the Schouten bracket of $\mathcal{H}$ with itself [15]. The simplest approach to verifying if an operator has vanishing Schouten bracket is the calculus of multi-vectors developed in [29] which will be adapted to the $\mathfrak{h}_\perp$-valued setting here.

First, on x-jet space J^∞, introduce a $\mathfrak{h}_\perp$-valued *vertical uni-vector* θ which is dual to $\mathrm{d}u$ regarded as a $\mathfrak{h}_\perp$-valued 1-form, in the sense that $-\langle \theta, \mathrm{d}u \rangle = 1$ where the pairing denotes the usual contraction between vectors and 1-forms, combined with their Cartan–Killing inner product on $\mathfrak{h}$. Then x-derivatives of θ are introduced similarly as duals to $\mathrm{d}u_x = D_x \mathrm{d}u$, $\mathrm{d}u_{xx} = D_x^2 \mathrm{d}u$, etc. via $\theta_x = D_x \theta$, $\theta_{xx} = D_x^2 \theta$, etc. such that $-\langle \theta_{kx}, \mathrm{d}u_{lx} \rangle = \delta_{kl}$ (with $k, l = 0, 1, 2, \ldots$). Formal sums of antisymmetric tensor products of these vertical uni-vectors will define *vertical multi-vectors*, for instance $\theta \wedge \theta$ and $\theta \wedge \theta \wedge \theta$ with respective values in $\Lambda_{\mathfrak{h}_\perp}^2$ and $\Lambda_{\mathfrak{h}_\perp}^3$. More particularly, a combined wedge product and Lie bracket of θ with itself yields a $\mathfrak{h}$-valued vertical bi-vector

$$[\theta, \theta] = [\theta, \theta]_\perp + [\theta, \theta]_\parallel \tag{92}$$

while the vertical tri-vector

$$[[\theta, \theta], \theta] = 0 \tag{93}$$

vanishes due to the Jacobi identity on $\mathfrak{h}$.

It is useful to let the basic uni-vector θ have the formal action of a vector field on $\mathfrak{h}_\perp$-valued functions $f(x, u, u_x, \ldots)$ on J^∞,

$$\delta_\theta f := \sum_{0 \leq k} \theta_{kx} \partial f / \partial u_{kx}. \tag{94}$$

Likewise, let θ act on differential operators $\mathcal{D} = \sum_{0 \le k} a_k D_x^k$ by a formal Lie derivative

$$\delta_\theta \mathcal{D} := \sum_{0 \le k} (\delta_\theta a_k) D_x^k \tag{95}$$

in terms of coefficients $a_k(x, u, u_x, \ldots)$ given by functions on J^∞. In particular, $\delta_\theta D_x = 0$. Finally, the action of θ can be naturally extended first to vertical uni-vectors θ via defining

$$\delta_\theta \theta := 0 \tag{96}$$

so $\delta_\theta \theta_{kx} = 0$, and then to vertical multi-vectors by letting δ_θ act as a derivation.

In this formalism the Schouten bracket of $\mathcal{H}$ with itself vanishes if and only if $\int \langle \theta \wedge \delta_{\mathcal{H}(\theta)} \mathcal{H}(\theta) \rangle \mathrm{d}x = 0$ where $\langle \cdot \wedge \cdot \rangle$ denotes the Cartan–Killing inner product combined with the wedge product of $\mathfrak{h}_\perp$-valued 1-forms on J^∞. Note, in particular, $\langle \theta \wedge \theta \rangle = 0$. An equivalent pointwise formulation is that $\langle \theta \wedge \delta_{\mathcal{H}(\theta)} \mathcal{H}(\theta) \rangle = D_x \Upsilon$ (for some scalar function Υ) or simply

$$\langle \theta \wedge \delta_{\mathcal{H}(\theta)} \mathcal{H}(\theta) \rangle \equiv 0 \text{ modulo a total } x\text{-derivative.} \tag{97}$$

This condition is equivalent to the Jacobi identity (82) holding for the Poisson bracket $\{\cdot, \cdot\}_{\mathcal{H}}$.

Proposition 4.8. *The linear operator* $\mathcal{H} = D_x + [u, \cdot]_\perp - [u, D_x^{-1}[u, \cdot]_\parallel]$ *in Theorem 4.5 has vanishing Schouten bracket and hence is cosymplectic.*

The following multi-vector identities will be used in the proof:

$$[A(\theta), B(\theta)] = [B(\theta), A(\theta)], \tag{98}$$
$$[A(\theta), [B(\theta), C(\theta)]] + cyclic = 0, \tag{99}$$

and

$$\langle A(\theta) \wedge B(\theta) \rangle = -\langle B(\theta) \wedge A(\theta) \rangle, \tag{100}$$
$$\langle A(\theta) \wedge [B(\theta), C(\theta)] \rangle = cyclic, \tag{101}$$

for linear operators A, B, C *from* $\mathfrak{h}_\perp$ *into* $\mathfrak{h}$*; also,*

$$\langle A(\theta) \wedge [u, \cdot] \rangle = -\langle u, [A(\theta), \cdot] \rangle = -\langle [u, A(\theta)] \wedge \cdot \rangle, \tag{102}$$
$$[[u, B(\theta)], A(\theta)] + [[u, A(\theta)], B(\theta)] = [u, [B(\theta), A(\theta)]], \tag{103}$$

and in particular

$$[[u, A(\theta)], A(\theta)] = \frac{1}{2}[u, [A(\theta), A(\theta)]]. \tag{104}$$

Proof. Consider

$$\delta_{\mathcal{H}(\theta)} \mathcal{H}(\theta) = [\mathcal{H}(\theta), \theta]_\perp - [\mathcal{H}(\theta), D_x^{-1}[u, \theta]_\parallel] - [u, D_x^{-1}[\mathcal{H}(\theta), \theta]_\parallel] \tag{105}$$

which expands out to a $\mathfrak{h}_\perp$-valued cubic polynomial in u. There is a single inhomogeneous term, $[D_x \theta, \theta]_\perp$. This yields in the l.h.s of (97):

$$\langle \theta \wedge [D_x \theta, \theta] \rangle = \frac{1}{3} D_x \langle \theta \wedge [\theta, \theta] \rangle \equiv 0 \tag{106}$$

by use of (101).

Next the linear terms in (105) are given by

$$-[D_x \theta, D_x^{-1}[u, \theta]_\parallel] - [u, D_x^{-1}[D_x \theta, \theta]_\parallel] + [[u, \theta]_\perp, \theta]_\perp. \tag{107}$$

In the l.h.s of (97) the first term of (107) yields:

$$-\langle \theta \wedge [D_x \theta, D_x^{-1}[u, \theta]_\parallel] \rangle = -\frac{1}{2} \langle D_x^{-1}[u, \theta]_\parallel \wedge D_x[\theta, \theta] \rangle \equiv \frac{1}{2} \langle [u, \theta]_\parallel \wedge [\theta, \theta] \rangle \tag{108}$$

from (98) and (101). Similarly the middle term of (107) simplifies:

$$-\langle \theta \wedge [u, D_x^{-1}[D_x\theta, \theta]_\parallel]\rangle = \langle u, [\theta, D_x^{-1}[D_x\theta, \theta]_\parallel]\rangle = -\frac{1}{2}\langle u, [\theta, [\theta, \theta]_\perp]\rangle = -\frac{1}{2}\langle [u, \theta] \wedge [\theta, \theta]_\perp\rangle \quad (109)$$

from (102), (92) and (93). The last term of (107) directly gives:

$$\langle \theta \wedge [[u, \theta]_\perp, \theta]\rangle = \langle [u, \theta]_\perp \wedge [\theta, \theta]\rangle. \quad (110)$$

Hence the linear terms in the l.h.s of (97) combine into

$$\equiv \frac{1}{2}\langle [u, \theta]_\parallel \wedge [\theta, \theta]\rangle + \frac{1}{2}\langle [u, \theta]_\perp \wedge [\theta, \theta]\rangle = \frac{1}{2}\langle [u, \theta] \wedge [\theta, \theta]\rangle = \frac{1}{2}\langle u, [\theta, [\theta, \theta]]\rangle = 0 \quad (111)$$

by (93) and (102).

The quadratic terms in (105) are given by

$$-[[u, \theta]_\perp, D_x^{-1}[u, \theta]_\parallel] - [[u, D_x^{-1}[u, \theta]_\parallel], \theta]_\perp - [u, D_x^{-1}[[u, \theta]_\perp, \theta]_\parallel]. \quad (112)$$

Write $A(\theta) = D_x^{-1}[u, \theta]_\parallel$ and combine the first and second terms in (112) by the Jacobi relation (99) to get $[[A(\theta), \theta], u]_\perp$. This yields in the l.h.s of (97):

$$\langle \theta \wedge [[A(\theta), \theta], u]\rangle = \langle A(\theta) \wedge [[u, \theta], \theta]\rangle \quad (113)$$

from (101) and (102). For the third term in (112), we use (102) to simplify the l.h.s of (97):

$$-\langle \theta \wedge [u, D_x^{-1}[[u, \theta]_\perp, \theta]_\parallel]\rangle = \langle [u, \theta] \wedge D_x^{-1}[[u, \theta]_\perp, \theta]_\parallel\rangle \equiv -\langle A(\theta) \wedge [[u, \theta]_\perp, \theta]\rangle. \quad (114)$$

This term now combines with the previous one, giving

$$\equiv \langle A(\theta) \wedge [[u, \theta]_\parallel, \theta]\rangle = \langle A(\theta) \wedge [D_x A(\theta), \theta]\rangle = \frac{1}{2}\langle \theta \wedge D_x[A(\theta), A(\theta)]_\perp\rangle \quad (115)$$

from (101). But the Lie bracket relation (66) implies $[A(\theta), A(\theta)]_\perp$ vanishes and hence so does the combined quadratic term (115) in the l.h.s of (97).

Finally, there are two cubic terms in (105):

$$[[u, A(\theta)], A(\theta)] + [u, D_x^{-1}[[u, A(\theta)], \theta]_\parallel] \quad (116)$$

where $A(\theta) = D_x^{-1}[u, \theta]_\parallel$ again. The second term of (116) simplifies in the l.h.s of (97):

$$\langle \theta \wedge [u, D_x^{-1}[[u, A(\theta)], \theta]_\parallel]\rangle = -\langle [u, \theta] \wedge D_x^{-1}[[u, A(\theta)], \theta]_\parallel\rangle$$
$$\equiv \langle A(\theta) \wedge [[u, A(\theta)], \theta]\rangle = \langle \theta \wedge [A(\theta), [u, A(\theta)]]\rangle \quad (117)$$

by (101) and (102). This term combines with the first term of (116) in the l.h.s of (97):

$$2\langle \theta \wedge [[u, A(\theta)], A(\theta)]\rangle = \langle \theta \wedge [u, [A(\theta), A(\theta)]_\parallel]\rangle = -\langle D_x A(\theta) \wedge [A(\theta), A(\theta)]\rangle$$

$$\equiv -\frac{1}{3}D_x\langle A(\theta) \wedge [A(\theta), A(\theta)]\rangle \equiv 0 \quad (118)$$

by means of (104), (101) and (102), in addition to $[A(\theta), A(\theta)]_\perp = 0$.

Thus the l.h.s of (97) vanishes modulo total x-derivatives, which completes the proof. $\square$

4.4. Proof of symplectic property for $\mathcal{J}$

A linear operator $\mathcal{J}$ that acts on $\mathfrak{h}_\perp$-valued functions in the x-jet space J^∞ of the flow variable u is a *symplectic operator* [15] iff the associated bilinear form (84) is skew and closed, namely $\omega(X, X) = 0$ and $d\omega(X, Y, Z) = \delta_X\omega(Y, Z) + \text{cyclic} = 0$ for all vector fields X, Y, Z on J^∞.

The conditions on ω have a straightforward pointwise formulation on J^∞ in terms of

$$\omega_{\mathcal{J}}(h_1^\perp, h_2^\perp) := -\langle h_1^\perp, \mathcal{J}(h_2^\perp)\rangle \quad (119)$$

where $h^\perp$ denotes a $\mathfrak{h}_\perp$-valued function. It will be convenient to work equivalently with $\mathfrak{m}_\perp$-valued functions $h_\perp = \mathrm{ad}(e_x)^{-1} h^\perp$ along with the operator $\mathcal{K}|_{\mathfrak{m}_\perp} = -\mathrm{ad}(e_x) \mathcal{J} \mathrm{ad}(e_x)$ as follows:

$$\omega_{\mathcal{J}}(h_1^\perp, h_2^\perp) = \langle \mathrm{ad}(e_x) h_{\perp 1}, \mathrm{ad}(e_x)^{-1} \mathcal{K}(h_{\perp 2}) \rangle = -\langle h_{\perp 1}, \mathcal{K}(h_{\perp 2}) \rangle. \tag{120}$$

Note, here, $\mathrm{ad}(e_x)$ is skew-adjoint with respect to the Cartan–Killing inner product on $\mathfrak{h}_\perp \oplus \mathfrak{m}_\perp$ as a consequence of the elementary identity

$$\langle \mathrm{ad}(e_x) A, B \rangle = \langle e_x, [A, B] \rangle = -\langle A, \mathrm{ad}(e_x) B \rangle \tag{121}$$

for functions A, B with values in $\mathfrak{h}_\perp \oplus \mathfrak{m}_\perp$.

To proceed, skew-symmetry of $\boldsymbol{\omega}$ will hold iff $\omega_{\mathcal{J}}$ is skew, which is clearly equivalent to $\mathcal{K}$ being a skew-adjoint operator:

$$\langle f, \mathcal{K}(f) \rangle \equiv 0 \text{ modulo a total } x\text{-derivative} \tag{122}$$

for all $\mathfrak{m}_\perp$-valued functions f on J^∞. This condition is formally identical to the skew-adjoint property established earlier for the operator $\mathcal{H} = \mathcal{K}|_{\mathfrak{h}_\perp}$ in Proposition 4.7, where, recall, $\mathcal{K}$ is the operator given in (74). As a result, the isomorphism $\mathfrak{m}_\perp \simeq \mathfrak{h}_\perp$ provided by $\mathrm{ad}(e_x)$ implies that the steps in the proof of Proposition 4.7 carry through with the function f now being $\mathfrak{m}_\perp$-valued instead of $\mathfrak{h}_\perp$-valued.

Proposition 4.9. *The linear operator* $\mathcal{J} = -\mathrm{ad}(e_x)^{-1}(D_x + [u, \cdot]_\perp - [u, D_x^{-1}[u, \cdot]_\parallel]) \, \mathrm{ad}(e_x)^{-1}$ *in Theorem 4.5 is skew-adjoint and hence the bilinear form* $\boldsymbol{\omega}(\cdot, \cdot) = \int \omega_{\mathcal{J}}(\cdot, \cdot) \mathrm{d}x$ *is skew-symmetric.*

Next, for a skew-adjoint operator $\mathcal{J}$, the closure property $d\boldsymbol{\omega} = 0$ is equivalent to having $\omega_{\mathcal{J}}$ satisfy $\delta_{h_1^\perp} \omega_{\mathcal{J}}(h_2^\perp, h_3^\perp) + \text{cyclic} = D_x \Upsilon$ for some scalar function Υ. This condition will hold iff:

$$\langle h_{\perp 3}, \delta_{h_1^\perp} \mathcal{K}(h_{\perp 2}) \rangle + \text{cyclic} = 0 \tag{123}$$

for all $\mathfrak{m}_\perp$-valued functions $h_{\perp i}$, with $h_i^\perp = \mathrm{ad}(e_x) h_{\perp i}$ ($i = 1, 2, 3$).

Hereafter it will be useful to let

$$(\cdot, \cdot) := [\mathrm{ad}(e_x) \cdot, \cdot] \tag{124}$$

which satisfies the following identities:

$$\langle A, (B, C) \rangle = -\langle C, (B, A) \rangle \tag{125}$$

and also

$$(A, B)_\parallel = (B, A)_\parallel, \tag{126}$$
$$(A, B)_\perp - (B, A)_\perp = \mathrm{ad}(e_x)[A, B], \tag{127}$$
$$\langle A, (B, C)_\perp \rangle + \text{cyclic} = 0, \tag{128}$$

for functions A, B, C with values in $\mathfrak{m}_\perp$. Note the latter two identities are readily established from the property that $\mathrm{ad}(e_x)$ acts as a derivation on $[\cdot, \cdot]$ in $\mathfrak{m}_\perp$ and annihilates $\mathfrak{m}_\parallel$.

Proposition 4.10. *The skew bilinear form* $\boldsymbol{\omega}(\cdot, \cdot) = \int \omega_{\mathcal{J}}(\cdot, \cdot) \mathrm{d}x$ *is closed and hence the linear operator* $\mathcal{J} = -\mathrm{ad}(e_x)^{-1}(D_x + [u, \cdot]_\perp - [u, D_x^{-1}[u, \cdot]_\parallel]) \mathrm{ad}(e_x)^{-1}$ *in Theorem 4.5 is symplectic.*

Proof. Consider, from (74) and (124),

$$\delta_{h_1^\perp} \mathcal{K}(h_{\perp 2}) = [h_1^\perp, h_{\perp 2}]_\perp - [h_1^\perp, D_x^{-1}[u, h_{\perp 2}]_\parallel] - [u, D_x^{-1}[h_1^\perp, h_{\perp 2}]_\parallel]$$

$$= (h_{\perp 1}, h_{\perp 2})_\perp - (h_{\perp 1}, D_x^{-1}[u, h_{\perp 2}]_\parallel) - [u, D_x^{-1}(h_{\perp 1}, h_{\perp 2})_\parallel] \tag{129}$$

which is a $\mathfrak{m}_\perp$-valued linear polynomial in u. The inhomogeneous term yields in the l.h.s of (123):

$$\langle h_{\perp 3}, (h_{\perp 1}, h_{\perp 2})_\perp \rangle + \text{cyclic} = 0 \tag{130}$$

from (128). Write $A(h_\perp) = D_x^{-1}[u, h_\perp]_\parallel$. Then in the l.h.s of (123) the middle term in (129) is given by:

$$-\langle h_{\perp 3}, (h_{\perp 1}, A(h_{\perp 2}))\rangle + \text{cyclic} = \langle A(h_{\perp 2}), (h_{\perp 1}, h_{\perp 3})_\parallel\rangle + \text{cyclic} \tag{131}$$

from (125). For the last term in (129), the l.h.s of (123) simplifies:

$$-\langle h_{\perp 3}, [u, D_x^{-1}(h_{\perp 1}, h_{\perp 2})_\parallel]\rangle + \text{cyclic} = \langle [u, h_{\perp 3}], D_x^{-1}(h_{\perp 1}, h_{\perp 2})_\parallel\rangle + \text{cyclic}$$
$$\equiv -\langle A(h_{\perp 3}), (h_{\perp 1}, h_{\perp 2})\rangle + \text{cyclic} = -\langle A(h_{\perp 2}), (h_{\perp 3}, h_{\perp 1})\rangle + \text{cyclic} \tag{132}$$

by cyclic rearrangement. Note $A(h_\perp)_\perp = 0$. Hence, from (126), the terms (131) and (132) cancel,

$$\equiv \langle A(h_{\perp 2}), (h_{\perp 1}, h_{\perp 3})_\parallel - (h_{\perp 3}, h_{\perp 1})_\parallel\rangle = 0 \tag{133}$$

which completes the proof. $\quad\square$

4.5. Proof of compatibility of $\mathcal{H}, \mathcal{J}$

Hamiltonian cosymplectic and symplectic operators $\mathcal{H}, \mathcal{J}$ are said to be *compatible (i.e. comprise a Hamiltonian pair)* if every linear combination $c_1\mathcal{H} + c_2\mathcal{J}^{-1}$ is a Hamiltonian operator. Equivalently [29,15], compatibility holds whenever $\mathcal{H}$ and $\mathcal{J}^{-1}$ have a vanishing Schouten bracket.

The latter condition can be formulated using the same calculus of vertical multi-vectors as employed in the proof of Proposition 4.8:

$$\langle \theta \wedge (\delta_{\mathcal{H}(\theta)}\mathcal{J}^{-1}(\theta) + \delta_{\mathcal{J}^{-1}(\theta)}\mathcal{H}(\theta))\rangle \equiv 0 \text{ modulo a total } x - \text{derivative} \tag{134}$$

where θ denotes the vertical uni-vector dual to du in the x-jet space J^∞ of u. It is possible to simplify the l.h.s. of (134) by noting $\delta_{\mathcal{H}(\theta)}\mathcal{J}^{-1} = -\mathcal{J}^{-1}(\delta_{\mathcal{H}(\theta)}\mathcal{J})\mathcal{J}^{-1}$ and then replacing θ by $\mathcal{J}(\theta)$ followed by using the skew-adjoint property of $\mathcal{J}$ to get

$$\langle \theta \wedge \delta_{\mathcal{H}(\mathcal{J}(\theta))}\mathcal{J}(\theta) + \mathcal{J}(\theta) \wedge (\delta_\theta\mathcal{H})\mathcal{J}(\theta)\rangle \equiv 0. \tag{135}$$

Now, as the uni-vector defined by $\mathcal{J}(\theta)$ obviously enjoys the same formal properties as θ itself, this establishes that (135) is equivalent to condition (134).

Lemma 4.11. *The Schouten bracket of $\mathcal{H}, \mathcal{J}$ vanishes iff (135) holds.*

For verifying if $\mathcal{H}, \mathcal{J}$ satisfy (135), the earlier multi-vector identities (98)–(102) will be useful, in addition to the following identities:

$$\langle A(\theta), (B(\theta), C(\theta))_\perp\rangle + \text{cyclic} = 0, \tag{136}$$
$$(A(\theta), B(\theta))_\parallel = -(B(\theta), A(\theta))_\parallel, \tag{137}$$

and in particular

$$(A(\theta), A(\theta))_\parallel = 0, \tag{138}$$

for linear operators A, B, C from $\mathfrak{h}_\perp$ into $\mathfrak{h}$. Here, $(\cdot, \cdot) = [\text{ad}(e_x)\cdot, \cdot]$ as before.

Theorem 4.12. *The cosymplectic and symplectic operators $\mathcal{H}, \mathcal{J}$ in Theorem 4.5 have vanishing Schouten bracket and hence comprise a Hamiltonian pair.*

Proof. It will be useful to introduce the uni-vectors $\tilde{\theta} := \text{ad}(e_x)^{-1}\theta$ and $\underset{\sim}{\theta} := \text{ad}(e_x)^{-1}\tilde{\theta}$, with values in $\mathfrak{m}_\perp$ and $\mathfrak{h}_\perp$, having the same formal properties as the $\mathfrak{h}_\perp$-valued uni-vector θ. (In particular, $\theta = \text{ad}(e_x)\tilde{\theta} = \text{ad}(e_x)^2 \underset{\sim}{\theta}$.) To begin, note $\langle \theta \wedge \delta_{\mathcal{H}(\mathcal{J}(\theta))}\mathcal{J}(\theta)\rangle = \langle \tilde{\theta} \wedge \delta_{\mathcal{H}(\mathcal{J}(\theta))}\mathcal{K}(\tilde{\theta})\rangle$ due to $\text{ad}(e_x)$ being skew-adjoint with respect to the Cartan–Killing inner product. Now consider

$$\delta_{\mathcal{H}(\mathcal{J}(\theta))}\mathcal{K}(\tilde{\theta}) = -[\mathcal{H}(\mathcal{J}(\theta)), D_x^{-1}[u, \tilde{\theta}]_\parallel] - [u, D_x^{-1}[\mathcal{H}(\mathcal{J}(\theta)), \tilde{\theta}]_\parallel] + [\mathcal{H}(\mathcal{J}(\theta)), \tilde{\theta}]_\perp \tag{139}$$

and write $A(\tilde{\theta}) := D_x^{-1}[u, \tilde{\theta}]_\parallel$. In the l.h.s. of (135), the first term in (139) yields

$$-\langle \tilde{\theta} \wedge [\mathcal{H}(\mathcal{J}(\theta)), A(\tilde{\theta})]\rangle = -\langle A(\tilde{\theta}) \wedge [\tilde{\theta}, \mathcal{H}(\mathcal{J}(\theta))]\rangle \tag{140}$$

and the second term in (139) simplifies:

$$-\langle \tilde{\theta} \wedge [u, D_x^{-1}[\mathcal{H}(\mathcal{J}(\theta)), \tilde{\theta}]_\parallel]\rangle \equiv -\langle A(\tilde{\theta}) \wedge [\tilde{\theta}, \mathcal{H}(\mathcal{J}(\theta))]\rangle. \tag{141}$$

Hence these two terms add to give, from (73),

$$-2\langle A(\tilde{\theta}) \wedge [\tilde{\theta}, D_x \mathcal{J}(\theta)]\rangle - 2\langle A(\tilde{\theta}) \wedge [[u, \mathcal{J}(\theta)]_\perp, \tilde{\theta}]\rangle + 2\langle A(\tilde{\theta}) \wedge [[u, B(\theta)], \tilde{\theta}]\rangle \tag{142}$$

where $B(\tilde{\theta}) := D_x^{-1}[u, \mathcal{J}(\theta)]_\parallel$. Next, in the l.h.s. of (135), the third term in (139) expands out to yield

$$\langle \tilde{\theta} \wedge [\mathcal{H}(\mathcal{J}(\theta)), \tilde{\theta}]\rangle = \langle [\tilde{\theta}, \tilde{\theta}] \wedge D_x \mathcal{J}(\theta)\rangle + \langle [\tilde{\theta}, \tilde{\theta}] \wedge [u, \mathcal{J}(\theta)]_\perp\rangle - \langle [\tilde{\theta}, \tilde{\theta}] \wedge [u, B(\theta)]\rangle. \tag{143}$$

Note the six terms given by (142) and (143) are linear in $J(\theta)$. Six similar terms come from $\langle \mathcal{J}(\theta) \wedge (\delta_\theta \mathcal{H})\mathcal{J}(\theta)\rangle$ as follows. First consider

$$(\delta_\theta \mathcal{H})\mathcal{J}(\theta) = -[\theta, D_x^{-1}[u, \mathcal{J}(\theta)]_\parallel] - [u, D_x^{-1}[\theta, \mathcal{J}(\theta)]_\parallel] + [\theta, \mathcal{J}(\theta)]_\perp. \tag{144}$$

In the l.h.s. of (135), the second term in (144) simplifies via (101)

$$-\langle \mathcal{J}(\theta) \wedge [u, D_x^{-1}[\theta, \mathcal{J}(\theta)]_\parallel]\rangle \equiv -\langle B(\theta) \wedge [\theta, \mathcal{J}(\theta)]\rangle \tag{145}$$

which then adds with the first term in (144), yielding by (73) and (137),

$$-2\langle B(\theta) \wedge [\theta, \mathcal{J}(\theta)]\rangle = -2\langle B(\theta) \wedge (\tilde{\theta}, \mathcal{J}(\theta))_\parallel\rangle = -2\langle B(\theta) \wedge [\mathcal{K}(\tilde{\theta}), \tilde{\theta}]\rangle. \tag{146}$$

This expands out to give three terms: first

$$-2\langle B(\theta) \wedge [D_x \tilde{\theta}, \tilde{\theta}]\rangle \equiv \langle [u, \mathcal{J}(\theta)]_\parallel \wedge [\tilde{\theta}, \tilde{\theta}]\rangle \tag{147}$$

by (98); next

$$-2\langle B(\theta) \wedge [[u, \tilde{\theta}]_\perp, \tilde{\theta}]\rangle = -2\langle B(\theta) \wedge [[u, \tilde{\theta}], \tilde{\theta}]_\parallel\rangle = -\langle B(\theta) \wedge [u, [\tilde{\theta}, \tilde{\theta}]]\rangle \tag{148}$$

by $[[u, \theta]_\parallel, \theta]_\parallel = 0$ from the Lie bracket relations (67) followed by (104); third

$$2\langle B(\theta) \wedge [[u, A(\tilde{\theta})], \tilde{\theta}]\rangle. \tag{149}$$

The last term remaining in (144) can be rearranged in the l.h.s. of (135):

$$\langle \mathcal{J}(\theta) \wedge [\theta, \mathcal{J}(\theta)]\rangle = \langle \mathcal{J}(\theta) \wedge (\tilde{\theta}, \mathcal{J}(\theta))_\perp\rangle$$
$$= -\langle \tilde{\theta} \wedge (\mathcal{J}(\theta), \mathcal{J}(\theta))_\perp\rangle - \langle \mathcal{J}(\theta) \wedge (\mathcal{J}(\theta), \tilde{\theta})_\perp\rangle = 2\langle \mathcal{J}(\theta) \wedge [\tilde{\theta}, \mathcal{K}(\tilde{\theta})]\rangle \tag{150}$$

via (101) and (136). This term expands out to give three terms:

$$\langle \mathcal{J}(\theta) \wedge D_x[\tilde{\theta}, \tilde{\theta}]\rangle - 2\langle \mathcal{J}(\theta) \wedge [\tilde{\theta}, [u, A(\tilde{\theta})]]\rangle + 2\langle \mathcal{J}(\theta) \wedge [\tilde{\theta}, [u, \tilde{\theta}]_\perp]\rangle \tag{151}$$

via (98).

Now the term (148) cancels the last term in (143) by (102), while the first terms in Eqs. (143) and (151) reduce to $D_x\langle \mathcal{J}(\theta) \wedge [\tilde{\theta}, \tilde{\theta}]\rangle \equiv 0$.

The remaining terms in (143) and (151), along with the three terms in (142) and the two terms given by (147) and (149), constitute a cubic polynomial in u with coefficients linear in $\mathcal{J}(\theta)$. The linear u-terms in this polynomial consist of:

$$2\langle \mathcal{J}(\theta) \wedge [\tilde{\theta}, [u, \tilde{\theta}]_\perp]\rangle = \langle \mathcal{J}(\theta) \wedge [u, [\tilde{\theta}, \tilde{\theta}]]\rangle - 2\langle D_x A(\tilde{\theta}) \wedge [\tilde{\theta}, \mathcal{J}(\theta)]\rangle \tag{152}$$

via (101) and (104); plus

$$\langle [\tilde{\theta}, \tilde{\theta}] \wedge [u, \mathcal{J}(\theta)]\rangle \tag{153}$$

and

$$-2\langle A(\tilde\theta) \wedge [\tilde\theta, D_x \mathcal{J}(\theta)]\rangle. \tag{154}$$

When combined, these four terms reduce to

$$\equiv 2\langle A(\tilde\theta) \wedge [D_x\tilde\theta, \mathcal{J}(\theta)]\rangle. \tag{155}$$

Next, the quadratic u-terms in the polynomial yield

$$-2\langle \mathcal{J}(\theta) \wedge [\tilde\theta, [u, A(\tilde\theta)]]\rangle - 2\langle A(\tilde\theta) \wedge [\tilde\theta, [u, \mathcal{J}(\theta)]_\perp]\rangle = -2\langle\tilde\theta \wedge [u, [A(\tilde\theta), \mathcal{J}(\theta)]]\rangle \tag{156}$$

via (101) and (103), with $[\tilde\theta, [u, \mathcal{J}(\theta)]_\parallel]_\parallel = 0$ by the Lie bracket relation (67). Similarly, the cubic terms combine to produce

$$2\langle [[u, A(\tilde\theta)], \theta] \wedge B(\tilde\theta)\rangle + 2\langle A(\tilde\theta) \wedge [[u, B(\tilde\theta)], \tilde\theta]\rangle = 2\langle\tilde\theta \wedge [u, [A(\tilde\theta), B(\theta)]]\rangle$$
$$= -2\langle D_x A(\tilde\theta) \wedge [A(\tilde\theta), B(\tilde\theta)]\rangle \equiv \langle [A(\tilde\theta), A(\tilde\theta)] \wedge [u, \mathcal{J}(\theta)]\rangle \tag{157}$$

where the middle step uses $[A(\tilde\theta), B(\tilde\theta)]_\perp = 0$ due to the Lie bracket relations (66).

To proceed, we make use of the identity

$$\mathcal{J}(\theta) = -\mathrm{ad}(e_x)^{-1}\mathcal{K}(\tilde\theta) = -D_x \underset{\sim}{\theta} - C(\tilde\theta) + [\tilde u, A(\tilde\theta)] \tag{158}$$

with $C(\tilde\theta) := \mathrm{ad}(e_x)^{-1}[u, \tilde\theta]_\perp$ and $\tilde u := \mathrm{ad}(e_x)^{-1}u$ where the last term in (158) arises from $[u, A(\tilde\theta)] = \mathrm{ad}(e_x)[\tilde u, A(\tilde\theta)]$ as $\mathrm{ad}(e_x)$ annihilates $A(\tilde\theta)$. The terms (155)–(157) expand out to become a quintic polynomial in u as follows. There is a single linear term, coming from (155):

$$-2\langle A(\tilde\theta) \wedge [D_x\tilde\theta, D_x \underset{\sim}{\theta}]\rangle = -2\langle A(\tilde\theta) \wedge (D_x \underset{\sim}{\theta}, D_x \underset{\sim}{\theta})_\parallel\rangle = 0 \tag{159}$$

by (138). Next, one quadratic term comes from (155):

$$-2\langle A(\tilde\theta) \wedge [D_x\tilde\theta, C(\tilde\theta)]\rangle = -2\langle A(\tilde\theta) \wedge (D_x \underset{\sim}{\theta}, C(\tilde\theta))_\parallel\rangle = 2\langle A(\tilde\theta) \wedge [[u, \tilde\theta]_\perp, D_x \underset{\sim}{\theta}]\rangle$$
$$= 2\langle D_x \underset{\sim}{\theta} \wedge [A(\tilde\theta), [u, \tilde\theta]]\rangle \tag{160}$$

via (101) and (138) combined with $[A(\tilde\theta), [u, \tilde\theta]_\parallel]_\perp = 0$ by the Lie bracket relations (66). This term (160) now cancels the quadratic term given by (156),

$$2\langle\tilde\theta \wedge [u, [A(\tilde\theta), D_x \underset{\sim}{\theta}]]\rangle = -2\langle [u, \tilde\theta] \wedge [A(\tilde\theta), D_x \underset{\sim}{\theta}]\rangle \tag{161}$$

through (101). We then have only cubic and higher degree terms remaining to consider. One cubic term comes from (156):

$$2\langle\tilde\theta \wedge [u, [A(\tilde\theta), C(\tilde\theta)]]\rangle = -2\langle A(\tilde\theta) \wedge [[u, \tilde\theta], C(\tilde\theta)]\rangle = -2\langle A(\tilde\theta) \wedge (C(\tilde\theta), C(\tilde\theta))_\parallel\rangle = 0 \tag{162}$$

by (138), with $[[u, \tilde\theta]_\parallel, C(\tilde\theta)]_\parallel = 0$ by the Lie bracket relations (67). Another cubic term arises from (155):

$$2\langle A(\tilde\theta) \wedge [D_x\tilde\theta, [\tilde u, A(\tilde\theta)]]\rangle = \langle D_x\tilde\theta \wedge [\tilde u, [A(\tilde\theta), A(\tilde\theta)]]\rangle = \langle [A(\tilde\theta), A(\tilde\theta)] \wedge (D_x \underset{\sim}{\theta}, \tilde u)_\parallel\rangle \tag{163}$$

via (101) and (104). A similar cubic term is given by (157):

$$\langle [A(\tilde\theta), A(\tilde\theta)] \wedge [u, D_x \underset{\sim}{\theta}]\rangle = -\langle [A(\tilde\theta), A(\tilde\theta)] \wedge (\tilde u, D_x \underset{\sim}{\theta})_\parallel\rangle. \tag{164}$$

Then (163) and (164) cancel by (126). There is a similar cancellation of quartic terms coming from (156) and (157):

$$-2\langle\tilde\theta \wedge [u, [A(\tilde\theta), [\tilde u, A(\tilde\theta)]]]\rangle = \langle [A(\tilde\theta), A(\tilde\theta)] \wedge [[u, \tilde\theta]_\perp, \tilde u]\rangle = \langle [A(\tilde\theta), A(\tilde\theta)] \wedge (C(\tilde\theta), \tilde u)_\parallel\rangle \tag{165}$$

and

$$-\langle [A(\tilde{\theta}), A(\tilde{\theta})] \wedge [u, C(\tilde{\theta})] \rangle = -\langle [A(\tilde{\theta}), A(\tilde{\theta})] \wedge (\tilde{u}, C(\tilde{\theta}))_{\parallel} \rangle. \tag{166}$$

As a result, the only term remaining is quintic, which comes from (157):

$$\langle [A(\tilde{\theta}), A(\tilde{\theta})] \wedge [u, [\tilde{u}, A(\tilde{\theta})]] \rangle. \tag{167}$$

We now employ the Jacobi identity combined with the identity

$$[u, [\tilde{u}, A(\tilde{\theta})]]_{\parallel} = (\tilde{u}, [\tilde{u}, A(\tilde{\theta})])_{\parallel} = ([\tilde{u}, A(\tilde{\theta})], \tilde{u})_{\parallel} = [[u, A(\tilde{\theta})], \tilde{u}]_{\parallel} \tag{168}$$

to get

$$[u, [\tilde{u}, A(\tilde{\theta})]]_{\parallel} = \frac{1}{2}[[u, \tilde{u}], A(\tilde{\theta})]_{\parallel}. \tag{169}$$

Hence (167) simplifies to

$$\frac{1}{2}\langle [A(\tilde{\theta}), A(\tilde{\theta})] \wedge [[u, \tilde{u}], A(\tilde{\theta})] \rangle = \frac{1}{2}\langle [u, \tilde{u}] \wedge [[A(\tilde{\theta}), A(\tilde{\theta})], A(\tilde{\theta})] \rangle = 0 \tag{170}$$

by (99). Consequently, all terms in the l.h.s. of (135) have been shown to vanish modulo a total x-derivative, thereby completing the proof.

5. Bi-Hamiltonian hierarchies of non-stretching curve flows and soliton equations

In any Riemannian symmetric space $M = G/H$, the frame structure equations of non-stretching curve flows $\gamma(t, x)$ geometrically encode a group-invariant bi-Hamiltonian structure for the induced flow on the components of the principal normal vector along the curve. This structure looks simplest as stated in Theorem 4.5 using an H-parallel moving frame formulation (cf. Definition 3.3) expressed in terms of a $\mathfrak{m}$-valued linear coframe e and its associated $\mathfrak{h}$-valued linear connection ω, obeying properties (i) to (iv) summarized in Lemma 4.1. Such a framing of $\gamma(t, x)$ arises directly from the $\mathfrak{g}$-valued zero-curvature Cartan connection $^{\mathfrak{g}}\omega = e + \omega$ in the Klein geometry associated with G viewed as an H-bundle over M, where $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ is the corresponding Lie algebra decomposition of G and $\mathfrak{m} \simeq_G T_x M$. In particular, e and ω provide a soldering of the Klein geometry of G/H onto the Riemannian geometry of M, such that the frame structure equations coincide with the Cartan equations for torsion and curvature (cf Theorem 2.3 and Corollary 2.4) of the Riemannian connection ∇_t, ∇_x on the two-dimensional surface of the flow in M.

The bi-Hamiltonian structure consists of compatible Hamiltonian (cosymplectic and symplectic) operators $\mathcal{H}$ and $\mathcal{J}$, which yield a (hereditary) recursion operator

$$\mathcal{R} = \mathcal{H}\mathcal{J} = -(\mathcal{K}\mathrm{ad}(e_x)^{-1})^2$$
$$= -(\mathrm{ad}(e_x)^{-1}D_x + [u, \mathrm{ad}(e_x)^{-1} \cdot]_{\perp} - [u, D_x^{-1}[u, \mathrm{ad}(e_x)^{-1} \cdot]_{\parallel}])^2 \tag{171}$$

with respect to the flow variable u. Due to the H-parallel property of e and ω, $e_x = e \rfloor \gamma_x$ is a fixed unit vector in $\mathfrak{m}$ representing the moving frame components of the tangent vector γ_x along the curve, while $u = \omega_x = \omega \rfloor \gamma_x$ is related to the moving frame components of the principal normal vector $\nabla_x \gamma_x$ along the curve through the corresponding $\mathfrak{m}_{\perp}$-valued expression $[u, e_x] = -\mathrm{ad}(e_x)u = e \rfloor \nabla_x \gamma_x$. Note $\mathrm{ad}(e_x)^{-1}$ gives an isomorphism between the perp spaces $\mathfrak{h}_{\perp}$ and $\mathfrak{m}_{\perp}$ of the centralizer subspaces $\mathfrak{h}_{\parallel}$ and $\mathfrak{m}_{\parallel}$ with respect to e_x in $\mathfrak{m} \oplus \mathfrak{h} = \mathfrak{g}$, with $\mathrm{ad}(e_x)\mathfrak{h}_{\parallel} = \mathrm{ad}(e_x)\mathfrak{m}_{\parallel} = 0$. Hence the linear operator $\mathcal{K}\mathrm{ad}(e_x)^{-1}$ maps $\mathfrak{h}_{\perp}$-valued functions into $\mathfrak{m}_{\perp}$-valued functions, and vice versa. Thus the adjoint of the recursion operator is given by

$$\mathcal{R}^* = \mathcal{J}\mathcal{H} = -(\mathrm{ad}(e_x)^{-1}\mathcal{K})^2$$
$$= -(\mathrm{ad}(e_x)^{-1}D_x + \mathrm{ad}(e_x)^{-1}[u, \cdot]_{\perp} - \mathrm{ad}(e_x)^{-1}[u, D_x^{-1}[u, \cdot]_{\parallel}])^2. \tag{172}$$

On the x-jet space $J^{\infty} = (x, u, u_x, u_{xx}, \ldots)$ of the flow variable u, the x-translation vector field $\partial/\partial x$ is an obvious symmetry of the operators $\mathcal{H}, \mathcal{J}$ and hence of the recursion operator $\mathcal{R}$. As a consequence, from general results due to Magri [23,24] on Hamiltonian recursion operators [29], $\mathcal{R}$ will generate a hierarchy of commuting Hamiltonian

vector fields with respect to the Poisson bracket, where the hierarchy starts with the evolutionary form of the vector field $\partial/\partial x$. Moreover, an associated hierarchy of involutive covector fields arises from the canonical pairing provided by the symplectic 2-form.

Theorem 5.1. *A commuting hierarchy of Hamiltonian vector fields $h_{(n)}^{\perp} \cdot \partial/\partial u$ is given by the $\mathfrak{h}_{\perp}$-valued functions*

$$h_{(n)}^{\perp} := \mathcal{R}^n(u_x), \quad (n = 0, 1, 2, \ldots) \tag{173}$$

satisfying the Poisson bracket relation

$$\delta_{h^{\perp}(n)} \cdot = \{\cdot, \mathfrak{H}^{(n)}\}_{\mathcal{H}} \tag{174}$$

for some Hamiltonian functionals $\mathfrak{H}^{(n)} = \int H^{(n)}(x, u, u_x, \ldots)$, with respect to the Hamiltonian operator $\mathcal{H}$. The variational derivative of each such functional yields a dual vector field $\varpi_{(n)}^{\perp} \cdot \mathrm{d}u$ given by the $\mathfrak{h}_{\perp}$-valued functions

$$\varpi_{(n)}^{\perp} := \delta H^{(n)}/\delta u = \mathcal{R}^{*n}(u), \quad (n = 0, 1, 2, \ldots) \tag{175}$$

forming a hierarchy of involutive variational covector fields. In particular, at the bottom of these hierarchies, $H^{(0)} = -\frac{1}{2}\langle u, u \rangle$ is the Hamiltonian for the vector field given by $h_{(0)}^{\perp} = u_x$ corresponding to the generator of x-translations in evolutionary form, whose dual covector field is given by $\varpi_{(0)}^{\perp} = u$.

Corollary 5.2. *These hierarchies are related by the Hamiltonian operators*

$$\varpi_{(n+1)}^{\perp} = \mathcal{J}(h_{(n)}^{\perp}), \quad h_{(n)}^{\perp} = \mathcal{H}(\varpi_{(n)}^{\perp}) \tag{176}$$

with $\langle \varpi_{(n)}^{\perp}, h_{(n)}^{\perp} \rangle \equiv 0$ modulo a total x-derivative, since $\mathcal{H}, \mathcal{J}$ are skew-adjoint.

The form of these operators under a scaling of the variables x and u shows that both hierarchies possess the mKdV scaling symmetry $x \to \lambda x, u \to \lambda^{-1} u$. Thereby $h_{(n)}^{\perp}$ and $H^{(n)}$ each have scaling weight $2 + 2n$, and $\varpi_{(n)}^{\perp}$ hence has scaling weight $1 + 2n$. From variational scaling methods in [2], it follows that $H^{(n)}$ can be expressed explicitly in terms of $h_{(n)}^{\perp}$ or $\varpi_{(n)}^{\perp}$:

$$(1 + 2n)H^{(n)} = -D_x^{-1}\langle u, h_{(n)}^{\perp} \rangle = -D_x^{-1}\langle u, \mathcal{H}(\varpi_{(n)}^{\perp}) \rangle. \tag{177}$$

The derivation of this formula is fairly simple. Let $\delta_s u = -x u_x - u = -D_x(xu)$ and $\delta_s H = -x D_x H - p H$ be the evolutionary form of the mKdV scaling symmetry generator on u and $H(x, u, u_x, \ldots)$, with respective scaling weights 1 and $p \neq 1$. Then we have

$$\delta_s H \equiv (1 - p)H \tag{178}$$

modulo a total x-derivative. On the other hand,

$$\delta_s H \equiv -\langle \delta_s u, \delta H/\delta u \rangle = \langle D_x(xu), \varpi^{\perp} \rangle \equiv -x\langle u, D_x \varpi^{\perp} \rangle = -x\langle u, \mathcal{H}(\varpi^{\perp}) \rangle, \tag{179}$$

where the last line uses the identity $\langle u, [u, A(\varpi^{\perp})] \rangle = \langle A(\varpi^{\perp}), [u, u] \rangle = 0$ holding for any linear operator A. Finally, the identity $\langle u, \mathcal{H}(\varpi^{\perp}) \rangle = D_x D_x^{-1}\langle u, \mathcal{H}(\varpi^{\perp}) \rangle$ yields $\delta_s H \equiv D_x^{-1}\langle u, \mathcal{H}(\varpi^{\perp}) \rangle$ modulo a total x-derivative, from which we obtain $H = (1 - p)^{-1} D_x^{-1}\langle u, \mathcal{H}(\varpi^{\perp}) \rangle$.

5.1. Group-invariant bi-Hamiltonian soliton equations

The entire hierarchy of vector fields (173) has two compatible Hamiltonian structures $\mathcal{H}$ and $\mathcal{J}^{-1}$:

$$h_{(n)}^{\perp} = \mathcal{H}(\delta H^{(n)}/\delta u) = \mathcal{J}^{-1}(\delta H^{(n+1)}/\delta u), \quad (n = 0, 1, 2, \ldots). \tag{180}$$

The operator $\mathcal{E} := \mathcal{R}\mathcal{H}$ provides an alternate Hamiltonian structure

$$h_{(n)}^{\perp} = \mathcal{E}(\delta H^{(n-1)}/\delta u) \quad (\text{for } n > 0), \tag{181}$$

with $\mathcal{H}$ and $\mathcal{E}$ comprising an explicit Hamiltonian pair (of cosymplectic operators). As a main result, this bi-Hamiltonian structure (180) produces a hierarchy of bi-Hamiltonian flows on u as given by linear combinations $h^\perp = h^\perp_{(n+1)} + h^\perp_{(n)}$ of the vector fields (173) according to the form of the flow equation (72).

Theorem 5.3. *The flow equation* (72) *on u yields a hierarchy of bi-Hamiltonian evolution equations*

$$u_t = \mathcal{H}(\delta H^{(n,1)}/\delta u) = \mathcal{J}^{-1}(\delta H^{(n+1,1)}/\delta u) \tag{182}$$

(hereafter called the $+1+n$ flow) for $n = 0, 1, 2, \ldots$, where $H^{(n,1)} := H^{(n)} + H^{(n-1)}$. Each of these $\mathfrak{h}_\perp$-valued evolution equations is invariant under the equivalence group $H^*_\parallel \subset H^*$ consisting of rigid (x-independent) linear transformations $\mathrm{Ad}(h^{-1})$ that preserve e_x.

These evolution equations (182) concretely describe group-invariant multicomponent soliton equations when u is expressed by real-valued components in an expansion with respect to any fixed basis (11) and (22) for the vector space $\mathfrak{h}_\perp \simeq \mathfrak{m}_\perp$. These components will naturally divide into irreducible representations u^α under the action of the group $H^*_\parallel$, corresponding to a direct sum decomposition $\mathfrak{h}_\perp = \oplus_\alpha \mathfrak{h}^\alpha_\perp$ where each subspace $\mathfrak{h}^\alpha_\perp$ is an eigenspace of $\mathrm{Ad}(H^*_\parallel)$. The specific algebraic structure of these eigenspaces (which will depend on the structure of the group $H^*_\parallel$) defines the algebraic content of the variables u^α: e.g. scalars, real vectors, complex vectors or vector pairs, scalar–vector pairs, etc. For instance, in the example $M = SO(N+1)/SO(N)$ studied in [3], the Lie algebra of $H^*_\parallel$ is $\mathfrak{h}_\parallel = \mathfrak{so}(N-1) \subset \mathfrak{h} = \mathfrak{so}(N)$, whence $\mathfrak{h}_\perp \simeq \mathbb{R}^{N-1} \simeq \mathfrak{m}_\perp$ with this vector space being irreducible (i.e. $\alpha = 1$). Thus, in this case, $u \in \mathbb{R}^{N-1}$ is algebraically identified as a single real vector variable, and the hierarchy of vector evolution equations (182) describes $SO(N-1)$-invariant multicomponent soliton equations for $u = (u^1, \ldots, u^{N-1})$.

In general the bi-Hamiltonian evolution equations (182) can be rewritten in terms of $\mathfrak{m}_\perp$-valued variables more geometrically related to the flow $\gamma(t, x)$ as follows:

$$N = \nabla_x \gamma_x \leftrightarrow \nu := -\mathrm{ad}(e_x)u \tag{183}$$

represents the principal normal vector along the curve;

$$(\gamma_t)_\perp \leftrightarrow h_\perp := \mathrm{ad}(e_x)^{-1}h^\perp, \quad (\gamma_t)_\parallel \leftrightarrow h_\parallel \tag{184}$$

represent the algebraically perpendicular and parallel parts of the flow vector;

$$\nabla_t \gamma_x \leftrightarrow \varpi_\perp := -\mathrm{ad}(e_x)\varpi^\perp = -\mathrm{ad}(e_x)(\varpi^\perp + \varpi^\parallel) \tag{185}$$

represents the principal normal vector relative to the flow. These geometrical variables inherit a natural Hamiltonian cosymplectic and symplectic structure from the results in Theorem 5.1 and Corollary 5.2 expressed with respect to the flow variable

$$v := \mathrm{ad}(e_x)^{-1}u = \mathcal{X}^{-1}(\nu), \quad \mathcal{X} = -\mathrm{ad}(e_x)^2 \tag{186}$$

in the following way. To begin, the induced flow equation takes the form

$$v_t = \tilde{\mathcal{H}}(\varpi_\perp) + h_\perp, \quad \varpi_\perp = \tilde{\mathcal{J}}(h_\perp) \tag{187}$$

where

$$\tilde{\mathcal{H}} = -\mathrm{ad}(e_x)^{-1}\mathcal{K}|_{\mathfrak{h}_\perp}\mathrm{ad}(e_x)^{-1}, \quad \tilde{\mathcal{J}} = \mathcal{K}|_{\mathfrak{m}_\perp} \tag{188}$$

are compatible Hamiltonian cosymplectic and symplectic operators given in terms of

$$\mathcal{K} = D_x + [\mathrm{ad}(e_x)v, \cdot]_\perp - [\mathrm{ad}(e_x)v, D_x^{-1}[\mathrm{ad}(e_x)v, \cdot]_\parallel]_\perp \tag{189}$$

from (74) and (186).

Proposition 5.4. *On the x-jet space of geometrical flows with respect to v, $h_\perp \cdot \partial/\partial v$ is a Hamiltonian vector field*

$$\delta_{h^\perp}(\cdot) = \{\cdot, \mathfrak{H}\}_{\tilde{\mathcal{H}}} = -\int \langle \delta(\cdot)/\delta v, \tilde{\mathcal{H}}(\delta\mathfrak{H}/\delta v)\rangle \mathrm{d}x \tag{190}$$

with respect to the Poisson bracket $\{\cdot,\cdot\}_{\tilde{\mathcal{H}}}$, and $\varpi_\perp \cdot \mathrm{d}v$ is a variational covector field

$$\varpi_\perp = \delta\mathfrak{F}/\delta v \tag{191}$$

satisfying the duality relation

$$(\varpi_\perp \cdot \mathrm{d}v) \rfloor \cdot \ = \boldsymbol{\omega}(\cdot\,, h_\perp \cdot \partial/\partial v)_{\tilde{\mathcal{J}}} = -\int \langle\,\cdot\,, \tilde{\mathcal{J}}(h_\perp)\rangle \mathrm{d}x \tag{192}$$

with respect to the symplectic 2-form $\boldsymbol{\omega}(\cdot,\cdot)_{\tilde{\mathcal{J}}}$.

Theorem 5.5. *There is a bi-Hamiltonian hierarchy of evolution equations on the principal normal components (183) in a H-parallel moving frame, represented by*

$$v_t = \tilde{\mathcal{H}}(\delta\tilde{H}^{(n,1)}/\delta v) = \tilde{\mathcal{J}}^{-1}(\delta\tilde{H}^{(n+1,1)}/\delta v), \quad (n = 0, 1, 2, \ldots) \tag{193}$$

with Hamiltonians $\tilde{H}^{(n,1)} := H^{(n)} + H^{(n-1)}$ expressed in terms of the $\mathfrak{m}_\perp$-valued variables $(v, v_x, \ldots)$ via (177) and (186). Correspondingly, the normal flow components

$$h_\perp^{(n)} = \tilde{\mathcal{R}}^n(v_x) = \tilde{\mathcal{H}}(\varpi_\perp^{(n)}), \quad (n = 0, 1, 2, \ldots) \tag{194}$$

and the principal normal components with respect to the normal flow

$$\varpi_\perp^{(n)} = \tilde{\mathcal{R}}^{*\,n}(\mathcal{X}(v)) = \tilde{\mathcal{J}}(h_\perp^{(n-1)}), \quad (n = 1, 2, \ldots) \tag{195}$$

represent hierarchies of commuting Hamiltonian vector fields $h_\perp^{(n)} \cdot \partial/\partial v$ and involutive variational covector fields $\varpi_\perp^{(n)} \cdot \mathrm{d}v$ generated by the recursion operator

$$\tilde{\mathcal{R}} = \tilde{\mathcal{H}}\tilde{\mathcal{J}} = -(\mathrm{ad}(e_x)^{-1}\mathcal{K})^2 \tag{196}$$

and its adjoint

$$\tilde{\mathcal{R}}^* = \tilde{\mathcal{J}}\tilde{\mathcal{H}} = -(\mathcal{K}\,\mathrm{ad}(e_x)^{-1})^2. \tag{197}$$

This version of the flow equation and its bi-Hamiltonian structure is closest to the notation used in [3,4]. Note the invariance group of the operators $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{J}}$ is $H_\parallel^* \subset H^* \simeq \mathrm{Ad}(H)$, and the scaling weights of $h_\perp^{(n)}$ and $\varpi_\perp^{(n)}$ are $2 + 2n$ and $1 + 2n$ under the mKdV scaling group $x \to \lambda x$, $v \to \lambda^{-1}v$.

Through the geometric correspondences (183) and (185), the evolution equations (193) produce a bi-Hamiltonian hierarchy of non-stretching curve flows in $M = G/H$. The Hamiltonians are related to the tangential part of the flow by

$$g(\gamma_t, \gamma_x) \leftrightarrow -\langle h_\parallel^{(n)}, e_x\rangle = (1 + 2n)H^{(n)} \tag{198}$$

arising from (70) and (177), where

$$h_\parallel = D_x^{-1}[v, \mathrm{ad}(e_x)h_\perp]_\parallel, \qquad H^{(n)} = -\frac{1}{1 + 2n}D_x^{-1}\langle v, \mathcal{X}(h_\perp^{(n)})\rangle. \tag{199}$$

Remarks.

(1) The infinitesimal invariance group of the hierarchy (193) can be identified with the Lie subalgebra $\mathfrak{h}_\parallel \subset \mathfrak{h}$ given by the centralizer of e_x in the Lie algebra of the group H.
(2) The bi-Hamiltonian structure (176) of the hierarchy depends on the constant unit vector e_x in $\mathfrak{m} \simeq \mathfrak{g}/\mathfrak{h}$ determined by the framing of the curve flow $\gamma(t, x)$.
(3) Two curve flows encode equivalent (isomorphic) bi-Hamiltonian structures iff e_x lies on the same orbit of the linear isotropy group H^* in $\mathfrak{m}$ for both flows, where all such orbits are parametrized by the unit-norm elements in the Weyl chamber $a_*(\mathfrak{m})$ (cf. Proposition 3.2) contained in a fixed maximal abelian subspace $a \subset \mathfrak{m}$.
(4) Without loss of generality, in a given equivalence class, a representative e_x can be chosen such that the linear map $\mathrm{ad}(e_x)^2$ is diagonal on $\mathfrak{m}_\perp$.

Consequently, every Riemannian symmetric space $M = G/H$ will possess distinct bi-Hamiltonian hierarchies of non-stretching curve flows characterized by the distinct unit-norm elements in a canonical Weyl chamber $a_*(\mathfrak{m}) \subset \mathfrak{m} \simeq \mathfrak{g}/\mathfrak{h} \simeq_G T_x M$. For the example $M = SO(N+1)/SO(N)$, it is well known that every maximal abelian subspace $a \subset \mathfrak{m} = \mathfrak{so}(N+1)/\mathfrak{so}(N) \simeq \mathbb{R}^N$ has $\dim a = 1$, whence the Weyl chamber $a_*(\mathfrak{m})$ has a single unit-norm element. As a result, up to equivalence, there is a single bi-Hamiltonian hierarchy of curve flows in $M = SO(N+1)/SO(N)$.

The equations of the curve flows on $\gamma(t, x)$ in a given hierarchy in $M = G/H$ can be readily derived in an explicit form from the geometric correspondences (183) and (185) by means of Proposition 4.4.

5.2. mKdV flows

The $+1$ flow on v will now be derived in detail. This will require working out the first higher order Hamiltonian structures in the hierarchy (194) and (195). At the bottom of the hierarchy, note

$$h_\perp^{(0)} = v_x, \tag{200}$$

$$H^{(0)} = -\langle h_\parallel^{(0)}, e_x \rangle = -D_x^{-1} \langle v, \mathcal{X}(h_\perp^{(0)}) \rangle = -\frac{1}{2} |\mathrm{ad}(e_x)v|^2, \tag{201}$$

$$\varpi_\perp^{(0)} = \delta H^{(0)}/\delta v = \mathcal{X}(v), \tag{202}$$

along with

$$h_\parallel^{(0)} = D_x^{-1} [v, \mathrm{ad}(e_x) h_\perp^{(0)}]_\parallel = -\frac{1}{2} [\mathrm{ad}(e_x)v, v]_\parallel, \tag{203}$$

where, recall, $\mathcal{X} = -\mathrm{ad}(e_x)^2$. The next order in the hierarchy is given by

$$\begin{aligned} \delta H^{(1)}/\delta v &= \varpi_\perp^{(1)} = \tilde{\mathcal{J}}(h_\perp^{(0)}) = \tilde{\mathcal{R}}^*(\mathcal{X}(v)) \\ &= v_{xx} + [\mathrm{ad}(e_x)v, v_x - \frac{1}{2}[\mathrm{ad}(e_x)v, v]_\parallel], \end{aligned} \tag{204}$$

$$\begin{aligned} h_\perp^{(1)} &= \tilde{\mathcal{H}}(\varpi_\perp^{(1)}) = \tilde{\mathcal{R}}(v_x) \\ &= \mathcal{X}^{-1}(v_{xxx}) + \mathcal{X}^{-1}([\mathrm{ad}(e_x)v, v_x - \frac{1}{2}[\mathrm{ad}(e_x)v, v]_\parallel]_\perp)_x \\ &\quad - \mathrm{ad}(e_x)^{-1}[\mathrm{ad}(e_x)v, \mathrm{ad}(e_x)^{-1}(v_{xx} + [\mathrm{ad}(e_x)v, v_x]_\perp) \\ &\quad - \frac{1}{2}[v, [\mathrm{ad}(e_x)v, v]_\parallel]] - [v, [v, v_x + \frac{1}{3}[\mathrm{ad}(e_x)v, v]_\perp]_\parallel], \end{aligned} \tag{205}$$

$$\begin{aligned} h_\parallel^{(1)} &= D_x^{-1}[v, \mathrm{ad}(e_x) h_\perp^{(1)}]_\parallel \\ &= [\mathrm{ad}(e_x)^{-1} v_{xx}, v]_\parallel - \frac{1}{2}[\mathrm{ad}(e_x)^{-1} v_x, v_x]_\parallel + [\mathrm{ad}(e_x)^{-1} v, [\mathrm{ad}(e_x)v, v_x]]_\parallel \\ &\quad + [v, [\mathrm{ad}(e_x)^{-1} v_x, \mathrm{ad}(e_x)v]]_\parallel + \frac{1}{8}[v, [v, 3[\mathrm{ad}(e_x)v, v]_\parallel + [\mathrm{ad}(e_x)v, v]_\perp]]_\parallel \\ &\quad - \frac{1}{8}[\mathrm{ad}(e_x)^{-1}[\mathrm{ad}(e_x)v, v]_\perp, [\mathrm{ad}(e_x)v, v]_\perp]_\parallel, \end{aligned} \tag{206}$$

$$\begin{aligned} H^{(1)} &= -\langle h_\parallel^{(1)}, e_x \rangle = -D_x^{-1} \langle v, \mathcal{X}(h_\perp^{(1)}) \rangle \\ &= -\frac{1}{2}|v_x|^2 + \frac{1}{3}\langle v_x, [\mathrm{ad}(e_x)v, v]_\perp \rangle + \frac{1}{8}|[\mathrm{ad}(e_x)v, v]_\parallel|^2. \end{aligned} \tag{207}$$

The derivation of (204) is simple: The local part of $\tilde{\mathcal{J}}$ on (200) yields

$$D_x h_\perp^{(0)} + [\mathrm{ad}(e_x)v, h_\perp^{(0)}]_\perp = v_{xx} + [\mathrm{ad}(e_x)v, v_x]_\perp, \tag{208}$$

while the nonlocal part of $\tilde{\mathcal{J}}$ applied to (200) reduces to

$$[\mathrm{ad}(e_x)v, h_\parallel^{(0)}] = -\frac{1}{2}[\mathrm{ad}(e_x)v, [\mathrm{ad}(e_x)v, v]_\parallel]. \tag{209}$$

For the derivation of (205), the following identity will be needed:

$$[A, (B, C)]_\| - [C, (B, A)]_\| = [B, (A, C) - (C, A)]_\| \tag{210}$$

which is a consequence of the Jacobi identity on the Lie bracket $[\cdot, \cdot]$ combined with properties (126) and (127) of the bracket $(\cdot, \cdot) := [\mathrm{ad}(e_x)\cdot, \cdot]$, holding for $\mathfrak{m}$-valued functions A, B, C. To proceed, it is convenient to split up the local and nonlocal parts of $\tilde{\mathcal{H}}$ on (202). The nonlocal part consists of, via (126),

$$-\mathrm{ad}(e_x)^{-1}[\mathrm{ad}(e_x)v, D_x^{-1}[\mathrm{ad}(e_x)v, \mathrm{ad}(e_x)^{-1}\varpi_\perp^{(1)}]_\|] = [v, D_x^{-1}[v, \varpi_\perp^{(1)}]_\|] \tag{211}$$

which yields three terms: First,

$$[v, D_x^{-1}[v, v_{xx}]_\|] = \mathrm{ad}(v)[v, v_x]_\| \tag{212}$$

follows from the elementary identity $D_x[v, v_x] = [v, v_{xx}]$. Second,

$$[v, D_x^{-1}[v, [\mathrm{ad}(e_x)v, v_x]_\perp]_\|] = \mathrm{ad}(v)D_x^{-1}[v, (v, v_x)]_\| = \frac{1}{3}\mathrm{ad}(v)[v, (v, v)]_\| \tag{213}$$

is obtained from the Lie bracket relation (67) followed by the identity $D_x[v, (v, v)]_\| = 3[v, (v, v_x)]_\|$ arising from (210). Third,

$$-\frac{1}{2}[v, D_x^{-1}[v, [\mathrm{ad}(e_x)v, [\mathrm{ad}(e_x)v, v]_\|]]_\|] = -\frac{1}{4}\mathrm{ad}(v)D_x^{-1}[(v, v)_\|, (v, v)]_\| = 0 \tag{214}$$

arises from the same Lie bracket relation combined with (210). Next the local parts of $\tilde{\mathcal{H}}$ on (202) are straightforwardly given by expanding out

$$\mathcal{X}^{-1}(D_x\varpi_\perp^{(1)}) \tag{215}$$

and

$$-\mathrm{ad}(e_x)^{-1}[\mathrm{ad}(e_x)v, \mathrm{ad}(e_x)^{-1}\varpi_\perp^{(1)}]. \tag{216}$$

Then (205) directly follows by combining these terms with the previous ones.

The derivation of (206) is similar.

Now the $+1$ flow is given by $v_t = h_\perp^{(1)} + h_\perp^{(0)}$ which is equivalent to the evolution equation (187) with $h_\perp = h_\perp^{(0)}$ and $\varpi_\perp = \varpi_\perp^{(1)}$. This equation can be written out in a fairly short explicit form by means of the brackets $[\cdot, \cdot]$ and $(\cdot, \cdot)$ together with the following linear maps

$$\mathrm{ad}(v)_\perp := [v, \cdot]_\perp, \quad \mathrm{ad}^\perp(v) := \mathrm{ad}(e_x)^{-1}[\mathrm{ad}(e_x)v, \cdot]_\perp \tag{217}$$

defined on the vector space $\mathfrak{m}_\perp \simeq \mathfrak{h}_\perp$.

Proposition 5.6.

$$v_t = v_x + \mathcal{X}^{-1}(v_{xxx}) - \mathcal{X}^{-1}\left(\frac{1}{2}(v, (v, v)_\|) - (v_x, v)_\perp\right)_x - (\mathcal{X}^{-1}(v_{xx}), v)_\perp$$

$$- \mathrm{ad}(v)_\perp[v, v_x + \frac{1}{3}(v, v)_\perp]_\| + \frac{1}{2}\mathrm{ad}^\perp(v)[v, (v, v)_\|] - \mathrm{ad}^\perp(v)^2 v_x \tag{218}$$

is the $+1$ flow in the hierarchy (193). Up to a convective term, this evolution equation describes a $H_\|^*$-invariant mKdV soliton equation on the $\mathfrak{m}_\perp$-valued flow variable v, with two compatible Hamiltonian structures

$$v_t - v_x = \tilde{\mathcal{H}}(\delta\mathfrak{H}/\delta v) = \tilde{\mathcal{E}}(\delta\mathfrak{E}/\delta v) \tag{219}$$

given by

$$\mathfrak{H} = \int\left(-\frac{1}{2}|v_x|^2 + \frac{1}{3}\langle v_x, (v, v)\rangle + \frac{1}{8}|(v, v)_\||^2\right)\mathrm{d}x, \quad \mathfrak{E} = -\int\langle v, \mathcal{X}(v)\rangle\mathrm{d}x \tag{220}$$

where

$$\tilde{\mathcal{E}} := \tilde{\mathcal{R}}\tilde{\mathcal{H}} = -(\mathrm{ad}(e_x)^{-1}\mathcal{K})^2. \tag{221}$$

The mKdV flow equation (218) corresponds to a geometrical motion of the curve $\gamma(t, x)$ obtained through the identifications (71), (183), (184) arising from $\mathfrak{m} \leftrightarrow T_\gamma M$ as defined via a H-parallel moving frame along $\gamma(t, x)$. Thus, in terms of the principal normal variable (183), the mKdV flow is given by

$$\begin{aligned} h_\perp &= \mathcal{X}^{-1}(v_x) = \mathcal{X}^{-1}(\mathcal{D}_x v - [u, v]) = \mathcal{X}^{-1}\mathcal{D}_x v - \mathcal{X}^{-1}[v, \mathcal{X}^{-1}[v, e_x]] \\ &= \mathcal{X}^{-1}\mathcal{D}_x v - (\mathcal{X}^{-1}\mathrm{ad}(v))^2 e_x \end{aligned} \tag{222}$$

and

$$\begin{aligned} h_\parallel &= \frac{1}{2}[\mathcal{X}^{-1}(v), \mathcal{X}^{-1}(\mathrm{ad}(e_x)v)]_\parallel = -\frac{1}{2}[\mathcal{X}^{-1}(v), [\mathcal{X}^{-1}(v), e_x]]_\parallel \\ &= -\frac{1}{2}(\mathrm{ad}(\mathcal{X}^{-1}(v))^2 e_x)_\parallel \end{aligned} \tag{223}$$

where, recall, $v \leftrightarrow N = \nabla_x \gamma_x, \mathcal{D}_x v \leftrightarrow \nabla_x N = \nabla_x^2 \gamma_x, e_x \leftrightarrow X = \gamma_x$. Also note

$$\mathcal{X} \leftrightarrow -\mathrm{ad}_x(X)^2 = -\mathrm{ad}(\gamma_x)^2 := \mathcal{X}_\gamma \tag{224}$$

which represents an invertible linear map on $T_\gamma M$. Hence

$$h_\perp + h_\parallel = e_t \leftrightarrow \mathcal{X}_\gamma^{-1}\nabla_x N - (\mathcal{X}_\gamma^{-1}\mathrm{ad}_x(N))^2 X - \frac{1}{2}(\mathrm{ad}_x(\mathcal{X}_\gamma^{-1}N)^2 X)_\parallel = \gamma_t \tag{225}$$

yields the following G-invariant geometrical equation of motion

$$\gamma_t = \mathcal{X}_\gamma^{-1}\nabla_x^2 \gamma_x - (\mathcal{X}_\gamma^{-1}\mathrm{ad}_x(\nabla_x \gamma_x))^2 \gamma_x - \frac{1}{2}(\mathrm{ad}_x(\mathcal{X}_\gamma^{-1}\nabla_x \gamma_x)^2 \gamma_x)_\parallel, \quad |\gamma_x|_g = 1 \tag{226}$$

which will be called a *non-stretching mKdV map* on $M = G/H$.

Remark. It is important to emphasize that the mKdV evolution equation (218) and the associated geometric map equation (226) are bi-Hamiltonian integrable systems that describe multicomponent generalizations of the scalar mKdV equation. In particular, the evolution equation (218) possesses a hierarchy of higher order commuting symmetries and higher order conserved densities corresponding to the vector fields (194) and Hamiltonians (199) for $n \geq 1$.

Both the mKdV map (226) and evolution equation (218) greatly simplify in the case when the linear map $\mathcal{X} = -\mathrm{ad}(e_x)^2$ is a multiple of the identity,

$$\mathcal{X} = \chi \mathrm{id}_\perp, \quad \chi = \mathrm{const} \tag{227}$$

on the vector spaces $\mathfrak{m}_\perp \simeq \mathfrak{h}_\perp$. From Proposition 2.1, note that (227) implies the Lie bracket relations

$$[\mathfrak{h}_\perp, \mathfrak{h}_\perp] \subseteq \mathfrak{h}_\parallel, \qquad [\mathfrak{m}_\perp, \mathfrak{m}_\perp] \subseteq \mathfrak{h}_\parallel, \qquad [\mathfrak{h}_\perp, \mathfrak{m}_\perp] \subseteq \mathfrak{m}_\parallel, \tag{228}$$

so consequently

$$[\cdot, \cdot]_\perp = (\cdot, \cdot)_\perp = 0, \quad \mathrm{ad}(v)_\perp \cdot = \mathrm{ad}^\perp(v)\cdot = 0, \quad \text{on } \mathfrak{m}_\perp \simeq \mathfrak{h}_\perp, \tag{229}$$

while

$$\mathrm{ad}^\perp(v)\cdot = \mathcal{X}^{-1}\mathrm{ad}(\mathcal{X}(v))_\perp \cdot = \mathrm{ad}(v)\cdot, \quad \text{on } \mathfrak{m}_\parallel. \tag{230}$$

This implies, in addition, the identity

$$(A, (B, C)) - (C, (B, A)) = \chi[B, [C, A]] \tag{231}$$

holding for functions A, B, C with values in $\mathfrak{m}_\perp \simeq \mathfrak{h}_\perp$.

As a result, when property (227) holds, the mKdV evolution equation (218) can be shown to reduce to

$$v_t - \chi v_x = v_{xxx} - \frac{3}{2}(v_x, (v, v)) = \tilde{\mathcal{H}}(\delta H^{(1)}/\delta v) = \tilde{\mathcal{E}}(\delta H^{(0)}/\delta v) \tag{232}$$

with a factor χ absorbed into the time derivative, and $H^{(0)} = -\frac{1}{2}|v|^2$, $H^{(1)} = -\frac{1}{2}|v_x|^2 + \frac{1}{8}|(v, v)|^2$, where the bi-Hamiltonian structure becomes

$$\tilde{\mathcal{H}} = \chi^{-1} D_x - [v, D_x^{-1}[v, \cdot\,]], \quad \tilde{\mathcal{J}} = D_x - (v, D_x^{-1}(v, \cdot\,)), \quad \tilde{\mathcal{E}} = \tilde{\mathcal{H}}\tilde{\mathcal{J}}\tilde{\mathcal{H}}, \tag{233}$$

and where the recursion operator is given by

$$\tilde{\mathcal{R}} = \tilde{\mathcal{H}}\tilde{\mathcal{J}} = \chi^{-1}\left(D_x^2 - \chi[v, [v, \cdot\,]] - (v, (v, \cdot\,)) + \chi[v, D_x^{-1}[v_x, \cdot\,]]\right.$$
$$\left. - (v_x, D_x^{-1}(v, \cdot\,)) + [v, D_x^{-1}[v, (v, (v, D_x^{-1}(v, \cdot\,)))]]\right) \tag{234}$$

(while the adjoint operator $\tilde{\mathcal{R}}^*$ is given by simply switching the brackets $\sqrt{\chi}[\cdot, \cdot]$ and $(\cdot, \cdot)$ in $\tilde{\mathcal{R}}$).

The mKdV map equation (226) similarly simplifies to

$$\gamma_t = \nabla_x^2 \gamma_x - \frac{3}{2}\chi^{-1}\mathrm{ad}_x(\nabla_x \gamma_x)^2 \gamma_x. \tag{235}$$

These results (232)–(235) encompass the two versions of mKdV vector evolution equations and curve flows in $M = SO(N + 1)/SO(N)$, $SU(N)/SO(N)$ derived in [3], as well as the their complex-valued generalizations obtained from $M = SO(N + 1)$, $SU(N)$ in [4]. In particular the scalar mKdV equation and its bi-Hamiltonian structure is given by (232)–(235) in the case $M = S^2$.

5.3. Remarks on higher order flows

Compared to the mKdV flow (218), the $+n$ flow in the hierarchy (193) for $n \geq 2$ is a mKdV evolution equation of higher order $2n + 1$ on the flow variable v. The corresponding curve flows on $\gamma(t, x)$ in $M = G/H$ are geometrically described by higher order versions of the non-stretching mKdV map equation (226). All of these flows and associated geometric map equations are bi-Hamiltonian integrable systems given by a group-invariant multicomponent generalization of the scalar mKdV soliton hierarchy coming from curve flows in two-dimensional constant curvature spaces, $M = S^2$ (cf. Section 1). Like in the scalar case, each flow (193) turns out to be an explicit local polynomial expression in v and its x derivatives, for which a general proof should be possible by applying the locality results established in [32].

There is a direct differential geometric interpretation of the flows (193), stemming from the geometrical meaning of the flow variable $v = \mathcal{X}^{-1}(e \rfloor N)$ given by the principal normal vector $N = \nabla_x \gamma_x$ along the curve γ in an H-parallel moving frame. Because of the H-parallel property (cf. Proposition 3.6), the components of v represent differential covariants of γ relative to the equivalence group $H_\parallel^* \subset H^*$ of the framing. More precisely, these components are invariantly determined by the curve γ up the covariant action of the (x-independent) group $H_\parallel^*$ in the vector space $\mathfrak{m}_\perp \subset \mathfrak{m} \simeq_G T_x M$ containing v. In contrast, for the case $M = S^2$, there is essentially a unique framing and the equivalence group $H_\parallel^*$ comprises only discrete reflections about the unit tangent vector γ_x in the tangent plane $T_x M \simeq \mathbb{R}^2$. In this case v represents a differential invariant of γ, namely the classical curvature invariant. The results in Theorem 5.5 thus show that, in a generalization from $M = S^2$ to symmetric spaces $M = G/H$, differential covariants of the curve γ give a proper setting for a flow variable that geometrically encodes a natural bi-Hamiltonian structure.

5.4. SG flows

In addition to all the local polynomial flows (193) in the hierarchy given in Theorem 5.5, there is a nonlocal, non-polynomial flow arising from the kernel of the recursion operator (171) as follows. Consider the vector field $h_\perp \cdot \partial/\partial v$

defined by

$$0 = \varpi_{\perp} = \tilde{\mathcal{J}}(h_{\perp}) \tag{236}$$

which thus gets mapped into $\tilde{\mathcal{R}}(h_{\perp}) = 0$ by the recursion operator. This will be called the -1 flow, with $h_{\perp}$ assigned zero scaling weight under the mKdV scaling group $x \to \lambda x$, $v \to \lambda^{-1} v$. Geometrically, from relation (185), $\varpi_{\perp} = e \rfloor \nabla_t \gamma_x = 0$ immediately determines the underlying curve flow on γ to be the G-invariant equation of motion

$$\nabla_t \gamma_x = 0, \qquad |\gamma_x|_g = 1 \tag{237}$$

which is recognized as being a *non-stretching wave map* on $M = G/H$, with t and x interpreted as light cone coordinates for the curve flow.

The corresponding evolution equation (187) induced on the flow variable v is given by $h_{\perp} = h_{\perp}^{(-1)}$ such that $\varpi = \tilde{\mathcal{J}}(h_{\perp}^{(-1)}) = 0$, so hence v satisfies the -1 flow equation

$$v_t = h_{\perp}^{(-1)}, \quad D_x h_{\perp}^{(-1)} + [\mathrm{ad}(e_x)v, h_{\perp}^{(-1)} - D_x^{-1}[\mathrm{ad}(e_x)v, h_{\perp}^{(-1)}]_{\|}]_{\perp} = 0 \tag{238}$$

describing a nonlocal flow. The structure of this equation looks simpler if we work in terms of the flow-vector variable $h_{\perp} + h_{\|} = e_t = e \rfloor \gamma_t$, and the x-connection variable $u = \mathrm{ad}(e_x)v = \omega_x = \omega \rfloor \gamma_x$, which obey the frame structure equations (68)–(70) in Lemma 4.2. Write

$$h^{(-1)} := h_{\perp}^{(-1)} + h_{\|}^{(-1)} \tag{239}$$

and note

$$\varpi^{\perp} + \varpi^{\|} = \omega_t = \omega \rfloor \gamma_t = 0 \tag{240}$$

where $\varpi^{\perp} = -\mathrm{ad}(e_x)\varpi_{\perp} = 0$ and $\varpi^{\|} = -D_x^{-1}[u, \varpi^{\perp}]_{\|} = 0$ due to $\varpi_{\perp} = 0$.

Proposition 5.7. *The -1 flow equation* (238) *is equivalent to the nonlocal evolution equation*

$$u_t = \mathrm{ad}(e_x)h^{(-1)} \tag{241}$$

with

$$D_x h^{(-1)} = -[u, h^{(-1)}], \tag{242}$$

whose geometrical content expresses the vanishing of the connection (240) *in the flow direction γ_t.*

This flow (241) and (242) possesses the conservation law

$$0 = D_x |h^{(-1)}|^2, \quad |h^{(-1)}|^2 := -\langle h_{\perp}^{(-1)}, h_{\perp}^{(-1)} \rangle - \langle h_{\|}^{(-1)}, h_{\|}^{(-1)} \rangle = |\gamma_t|_g^2, \tag{243}$$

geometrically corresponding to

$$0 = \nabla_x |\gamma_t|_g^2, \tag{244}$$

admitted by the wave map equation (237). Consequently, a conformal scaling of t can be used to put

$$1 = |h^{(-1)}| = |\gamma_t|_g \tag{245}$$

whence the -1 flow is equivalent to a flow with unit speed.

In the examples $M = SO(N+1)/SO(N)$, $SU(N)/SO(N)$ and $M = SO(N+1)$, $SU(N)$, the conformally scaled -1 flow given by equations (241)–(245) was shown [3,4] to admit an algebraic reduction yielding a hyperbolic equation of local non-polynomial form $v_{tx} = f(v, v_t)$ on the flow variable $v = \mathrm{ad}(e_x)^{-1}u$. Here a related hyperbolic equation will be derived that directly generalizes the SG soliton equation from the scalar case $M = S^2$ to $M = G/H$, without any reductions being introduced.

To proceed, observe that Eq. (242) has the geometric formulation

$$\mathcal{D}_x h^{(-1)} = 0 \tag{246}$$

where

$$\mathcal{D}_x = D_x + [u, \cdot] \tag{247}$$

is a covariant derivative operator (cf. Proposition 4.4) associated with the x-connection in the flow. Now consider a local gauge transformation on the x-connection, given by a smooth function $q : \gamma \subset M = G/H \rightarrow H^* \simeq \text{Ad}(H)$, such that the operator (247) is flattened, $\mathcal{D}_x \rightarrow q \mathcal{D}_x q^{-1} = D_x$. This has the geometrical meaning that the transformed x-connection vanishes

$$u \rightarrow quq^{-1} - q_x q^{-1} := \tilde{u} = 0 \tag{248}$$

giving a linear differential equation $q_x = qu$ that determines q in terms of u. For any such q, it then follows that the transformed flow vector is constant

$$h^{(-1)} \rightarrow q h^{(-1)} q^{-1} := \tilde{h}, \qquad D_x \tilde{h} = 0 \tag{249}$$

where $\tilde{h}$ is some constant $\mathfrak{m}$-valued vector with unit norm due to property (245).

As a consequence, equation (246) is solved by

$$u = q^{-1} q_x, \qquad h^{(-1)} = q^{-1} \tilde{h} q \tag{250}$$

in terms of a H^*-valued potential q. Substitution of these expressions into the -1 flow equation (241) now leads to a main result.

Theorem 5.8. *The -1 flow on $v = \text{ad}(e_x)^{-1} u = \text{ad}(e_x)^{-1}(q^{-1} q_x)$ is given by a hyperbolic equation*

$$q_{tx} - q_t q^{-1} q_x = q[e_x, q^{-1} \tilde{h} q] = [q e_x q^{-1}, \tilde{h}] q \tag{251}$$

for the H^*-valued potential q, with $D_x \tilde{h} = 0$ and $|\tilde{h}|_{\mathfrak{m}} = 1$.

This flow depends on the pair of constant unit vectors $e_x, \tilde{h}$ in $\mathfrak{m}$. Without loss of generality, under a rigid (x-independent) transformation $\tilde{h} \rightarrow \tilde{q} \tilde{h} \tilde{q}^{-1}$ if necessary (which preserves (250)), the vector $\tilde{h}$ can be taken to belong to the same Weyl chamber $a_*(\mathfrak{m})$ that contains the vector e_x in a fixed maximal abelian subspace $\mathfrak{a} \subset \mathfrak{m}$, whereby the flow (251) is parametrized by unit-norm elements in the Weyl chamber $a_*(\mathfrak{m})$ in $\mathfrak{m}$ (cf. Proposition 3.2). Note the invariance group of the flow will be $H_{\parallel}^*$ iff $\tilde{h} = e_x$.

A natural Hamiltonian structure can be derived for the hyperbolic flow equation (251) starting from the scaling formula (199) extended to $n = -1$. Let

$$H^{(-1)} := \langle e_x, h_{\parallel}^{(-1)} \rangle = \langle e_x, h^{(-1)} \rangle = \langle e_x, q^{-1} \tilde{h} q \rangle. \tag{252}$$

Its variational derivative with respect to q is defined by $\delta H^{(-1)} = \text{tr}(\delta q (\delta H^{(-1)}/\delta q)) = -\langle q^{-1} \delta q, (\delta H^{(-1)}/\delta q) q \rangle$, which yields

$$-(\delta H^{(-1)}/\delta q) q = [e_x, q^{-1} \tilde{h} q] = \text{ad}(e_x) h^{(-1)}. \tag{253}$$

Theorem 5.9. *The hyperbolic flow equation (251) has the Hamiltonian structure*

$$q^{-1}(q_{tx} - q_t q^{-1} q_x) q^{-1} = -\delta H^{(-1)}/\delta q \tag{254}$$

or equivalently

$$q_t q^{-1} = -D_x^{-1}(q(\delta H^{(-1)}/\delta q)) \tag{255}$$

as an evolutionary flow, where D_x^{-1} is a Hamiltonian operator on the x-jet space of $\mathfrak{h}$-valued functions.

This Hamiltonian structure has an equivalent formulation in terms of the flow variable $v = \mathrm{ad}(e_x)^{-1}u$ by means of the variational relations

$$\delta H^{(-1)} = -\langle \delta u, \delta H^{(-1)}/\delta u \rangle = -\langle \delta v, \delta H^{(-1)}/\delta v \rangle, \tag{256}$$

$$\delta u = \mathrm{ad}(e_x)\delta v = \delta(q^{-1}q_x) = \mathcal{D}_x(q^{-1}\delta q), \tag{257}$$

combined with the variational derivative (253), where $\delta H^{(-1)}/\delta u$ and $\delta H^{(-1)}/\delta v$ are $\mathfrak{h}_{\perp}$- and $\mathfrak{m}_{\perp}$-valued respectively. Note, since $(\delta u)_{\parallel} = 0$, the relation (257) implies $(q^{-1}\delta q)_{\parallel} = -D_x^{-1}[u, (q^{-1}\delta q)_{\perp}]_{\parallel}$ and hence

$$\delta u = D_x(q^{-1}\delta q)_{\perp} + [u, (q^{-1}\delta q)_{\perp} - D_x^{-1}[u, (q^{-1}\delta q)_{\perp}]_{\parallel}]_{\perp} = \mathcal{K}(q^{-1}\delta q)_{\perp} \tag{258}$$

where $\mathcal{K}$ is the operator (74).

Proposition 5.10. *The -1 flow equation (241) and (242) has the Hamiltonian structure*

$$u_t = \mathcal{H}(\delta H^{(-1)}/\delta u) \tag{259}$$

where $H^{(-1)} = \langle e_x, q^{-1}\tilde{h}q \rangle$ is a nonlocal function of u determined by $q_x = qu$. Equivalently, the -1 flow equation (238) on $v = \mathrm{ad}(e_x)^{-1}u$ is given by

$$v_t = \tilde{\mathcal{H}}(\delta H^{(-1)}/\delta v) \tag{260}$$

with q expressed in terms of v through $q_x = q[e_x, v]$.

As a consequence, the -1 flow on v thus defines a nonlocal Hamiltonian vector field

$$h_{\perp}^{(-1)} := (q^{-1}\tilde{h}q)_{\perp} = \tilde{\mathcal{H}}(\delta H^{(-1)}/\delta v) \tag{261}$$

and likewise

$$h_{(-1)}^{\perp} := \mathrm{ad}(e_x)h_{\perp}^{(-1)} = [e_x, q^{-1}\tilde{h}q] = \mathcal{H}(\delta H^{(-1)}/\delta u) \tag{262}$$

is a corresponding nonlocal Hamiltonian vector field given by the -1 flow on u.

Remarks. The group-valued hyperbolic equation (254) provides a multicomponent generalization of the scalar SG equation along with its Hamiltonian structure, arising from curve flows (237) in $M = G/H$. In particular, for $M = S^2$ if $H^* = SO(2) \simeq U(1)$ is viewed as the group of abelian unitary matrices then $q = \exp(i\theta)$ can be identified as a $U(1)$-valued potential satisfying the SG equation $\theta_{tx} = \sin(\theta - \tilde{\theta})$ with the Hamiltonian $H^{(-1)} = i\Im(\exp(i\theta)\exp(-i\tilde{\theta}))$, $\tilde{\theta} =$ constant. From Theorems 5.8 and 5.9 and Proposition 5.10, the generalization of this soliton equation in the multicomponent (nonabelian) case describes an integrable SG system for which the associated -1 flow in nonlocal evolutionary form (255) possesses a hierarchy of higher order commuting symmetries given by the vector fields (194) and higher order conserved densities given by the Hamiltonians (199) for $n \geq 0$. Moreover, these hierarchies can be expressed directly in terms of the potential q via the relation $v = \mathrm{ad}(e_x)^{-1}u = \mathrm{ad}(e_x)^{-1}(q^{-1}q_x)$.

6. Concluding remarks

The main results in this paper on bi-Hamiltonian hierarchies of geometric curve flows can be directly extended to flat Riemannian symmetric spaces $\mathfrak{m} = \mathfrak{g}/\mathfrak{h} \simeq \mathbb{R}^n$ viewed as infinitesimal Klein geometries. One significant geometrical difference will be that both the curvature and torsion of the Riemannian connection ∇ will vanish in the Cartan structure equations for the framing of the curve flows $\gamma(t, x)$ in $\mathbb{R}^n$. This change does not affect the encoding of bi-Hamiltonian operators, which will be the same Hamiltonian cosymplectic and symplectic operators as derived in curved symmetric spaces $M = G/H$, but it does lead to a simpler recursion structure for the resulting integrable hierarchy of group-invariant flow equations induced on the moving frame components of the principal normal vector along the curves. In particular, the higher order mKdV flows in each hierarchy for $M = G/H$ (when ∇ has non-zero curvature) are linear combinations of the ones arising in $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ (when ∇ is flat).

A further extension of the main results in the present paper arises for Riemannian symmetric spaces $M = G/H \simeq K$ given by any compact semisimple Lie group K, where $G = K \times K$ is a product group and $H \simeq K$ is a diagonal

subgroup. Since *M* here is a group manifold, there turns out to be an algebraic Hamiltonian operator that is compatible with the cosymplectic and symplectic operators encoded in the Cartan structure equations for the framing of curve flows $\gamma(t, x)$. This leads to an enlarged bi-Hamiltonian hierarchy of geometric curve flows, containing group-invariant NLS flows in addition to mKdV flows. The same result extends to the flat spaces $\mathfrak{m} = \mathfrak{g}/\mathfrak{h} \simeq \mathfrak{k}$ for any compact semisimple Lie algebra $\mathfrak{k}$ (viewed as an infinitesimal Klein geometry). Details of this will presented in a forthcoming paper.

Enlarged bi-Hamiltonian hierarchies of geometric curve flows will likewise arise for other symmetric spaces that carry extra algebraic structure, such as hermitian spaces or quaternion spaces. All such spaces are well known from Cartan's classification [19].

Earlier work [3,4] studying the four symmetric spaces $SO(N + 1)/SO(N)$, $SU(N)/SO(N)$, $SO(N + 1)$, $SU(N)$ geometrically accounted for all examples of $O(N - 1)$-invariant and $U(N - 1)$-invariant multicomponent mKdV and SG equations that have been found in symmetry–integrability classifications of vector evolution equations [34] and vector hyperbolic equations [6]. Building on these examples, a subsequent paper will be devoted to a full classification of further examples of group-invariant soliton equations derived from geometric curve flows in all classical (non-exceptional) symmetric spaces.

Finally, in light of work done in [10,11] deriving numerous scalar soliton equations from geometric curve flows in planar Klein geometries $G/H \simeq \mathbb{R}^2$, a natural interesting question is whether the present analysis using moving parallel frames for curve flows in Riemannian symmetric spaces can be broadly extended to general homogeneous spaces $M = G/H$. Such a generalization would encompass non-Riemannian Klein geometries and, accordingly, should yield group-invariant soliton equations (along with their bi-Hamiltonian structure) of other than mKdV and SG type.

Acknowledgments

References

 [1] S.C. Anco, J. Phys. A: Math. Gen. 39 (2006) 2043–2072.
 [2] S.C. Anco, J. Phys. A: Math. Gen. 36 (2003) 8623–8638.
 [3] S.C. Anco, SIGMA 2 (044) (2006) 18.
 [4] S.C. Anco, IMA Volumes in Mathematics and its Applications, vol. 144, Symmetries and Overdetermined Systems of Partial Differential Equations, 2007, pp. 223–250.
 [5] S.C. Anco, J.-P. Wang, 2007, in preparation.
 [6] S. Anco, T. Wolf, J. Nonlinear Math. Phys. 12 (2005) 13–31;
     S. Anco, T. Wolf, J. Nonlinear Math. Phys. 12 (2005) 607–608.
 [7] C. Athorne, A. Fordy, J. Phys. A: Math. Gen. 20 (1987) 1377–1386.
 [8] C. Athorne, J. Phys. A: Math. Gen. 21 (1988) 4549–4556.
 [9] R. Bishop, Amer. Math. Monthly 82 (1975) 246–251.
[10] K.-S. Chou, C. Qu, Physica D 162 (2002) 9–33.
[11] K.-S. Chou, C. Qu, Chaos Solitons Fractals 14 (2002) 29–44.
[12] K.-S. Chou, C. Qu, J. Nonlinear Sci. 13 (2003) 487–517.
[13] K.-S. Chou, C. Qu, Chaos Solitons Fractals 19 (2004) 47–53.
[14] A. Doliwa, P.M. Santini, Phys. Lett. A 185 (1994) 373–384.
[15] I. Dorfman, Dirac Structures and Integrability of Nonlinear Evolution Equations, Wiley, 1993.
[16] R.E. Goldstein, D.M. Petrich, Phys. Rev. Lett. 67 (1991) 3203–3206.
[17] H. Guggenheimer, Differential Geometry, McGraw-Hill, 1963.
[18] M. Gurses, Phys. Lett. A 241 (1998) 329–334.
[19] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Amer. Math. Soc., Providence, 2001.
[20] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, Vols. I and II, Wiley, 1969.
[21] G.L. Lamb Jr., J. Math. Phys. 18 (1977) 1654–1661.
[22] J. Langer, R. Perline, Phys. Lett. A 239 (1998) 36–40.
[23] F. Magri, J. Math. Phys. 19 (1978) 1156–1162.
[24] F. Magri, in: M. Boiti, F. Pempinelli, G. Soliani (Eds.), Nonlinear Evolution Equations and Dynamical Systems, in: Lecture Notes in Physics, vol. 120, Springer-Verlag, 1980.

[25] G. Mari Beffa, Bull. Soc. Math. France 127 (3) (1999) 363–391.
[26] G. Mari Beffa, Poisson Geometry, Polish Acad. Sci. Warsaw, 1998, pp. 187–196.
[27] G. Mari Beffa, J. Sanders, J.-P. Wang, J. Nonlinear Sci. 12 (2002) 143–167.
[28] K. Nakayama, H. Segur, M. Waditi, Phys. Rev. Lett. 69 (1992) 2603–2606.
[29] P.J. Olver, Applications of Lie Groups to Differential Equations, Springer, New York, 1986.
[30] J. Sanders, J.-P. Wang, Moscow Math. J. 3 (2003) 1369–1393.
[31] J. Sanders, J.-P. Wang, J. Difference Equ. Appl. 12 (2006) 983–995.
[32] A. Sergyeyev, J. Phys. A: Math. Gen. 38 (2005) 3397–3407.
[33] R.W. Sharpe, Differential Geometry, Springer-Verlag, New York, 1997.
[34] V.V. Sokolov, T. Wolf, J. Phys. A: Math. and Gen. 34 (2001) 11139–11148.
[35] J.-P. Wang, in: S. Abenda, G. Gaeta, S. Walcher (Eds.), SPT 2002: Symmetry and Perturbation Theory (Cala Gonone), World Scientific, River Edge, NJ, 2002, pp. 276–283.
[36] E. Asadi, J. Sanders, in: G. Gaeta, R. Vitolo, S. Walcher (Eds.), Symmetry and Perturbation Theory (Proceedings of the International Conference on SPT 2007), 2007 (in press).